5.2 Exercise 8

8. $xy'' + y' + xy = 0 \quad ; \quad x_0 = 1$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$xy'' + y' + xy$$

$$= (x-1)y'' + y'' + y' + (x-1)y + y$$

$$= (x-1)\cdot\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$+ (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$+ \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \underbrace{\sum_{n=1}^{\infty} (n+1)(n) a_{n+1} (x-1)^n}_{\text{shift } n \text{ by } 1} + \underbrace{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n}_{\text{shift } n \text{ by } 2} + \underbrace{\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n}_{\text{shift } n \text{ by } 1}$$

$$+ \underbrace{\sum_{n=1}^{\infty} a_{n-1} (x-1)^n}_{\text{shift } n \text{ by } (-1)} + \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \underbrace{\underbrace{\left[2a_2 + a_1 + a_0 \right]}_{\text{(All } n=0 \text{ terms)}}}_{=0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+1)n a_{n+1} + (n+2)(n+1) a_{n+2} + (n+1) a_{n+1} + a_{n-1} + a_n \right]}_{=0} (x-1)^n$$

$$\Rightarrow \quad 2a_2 + a_1 + a_0 = 0 \qquad (n+2)(n+1) a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1} = 0$$

(a) $$\boxed{a_2 = -\frac{a_1 + a_0}{2} \qquad a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n+1)}}$$

$n = 1, 2, 3, \ldots$

5.2

#8. (continued)

$$a_2 = -\frac{a_1 + a_0}{2} \qquad a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

(b) In order to make this more manageable, pick values for a_0 and a_1 now.

y_1: $a_0 = 1, \quad a_1 = 0$

$$a_2 = -\frac{1}{2}$$

$$(n=1) \quad a_3 = -\frac{4a_2 + \cancel{a_1} + a_0}{(3)(2)} = -\frac{4(-\frac{1}{2}) + 1}{6} = -\frac{-2+1}{6} = \frac{1}{6}$$

$$(n=2) \quad a_4 = -\frac{9a_3 + a_2 + \cancel{a_1}}{(4)(3)} = -\frac{9(\frac{1}{6}) + (-\frac{1}{2})}{12} = -\frac{\frac{3}{2} - \frac{1}{2}}{12} = -\frac{1}{12}$$

$$\boxed{y_1 = \underbrace{1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3 - \tfrac{1}{12}(x-1)^4}_{\text{First four non-zero terms}} + \cdots}$$

y_2: $a_0 = 0, \quad a_1 = 1$

$$a_2 = -\frac{0+1}{2} = -\frac{1}{2}$$

$$(n=1) \quad a_3 = -\frac{(4)a_2 + a_1 + \cancel{a_0}}{(3)(2)} = -\frac{4(-\frac{1}{2}) + 1}{6} = -\frac{-2+1}{6} = \frac{1}{6}$$

$$(n=2) \quad a_4 = -\frac{9a_3 + a_2 + a_1}{(4)(3)} = -\frac{9(\frac{1}{6}) - \frac{1}{2} + 1}{12} = -\frac{\frac{3}{2} - \frac{1}{2} + 1}{12} = -\frac{2}{12} = -\frac{1}{6}$$

$$\boxed{y_2 = \underbrace{(1)(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3 - \tfrac{1}{6}(x-1)^4}_{\text{First four non-zero terms}} + \cdots}$$

(c) $$W(y_1, y_2)(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

(d) Finding the general term in these solutions seems to be a lost cause.