Example (Exercise 5.2.5 in book)

$$f(y;\theta) = \frac{y^3 e^{-y/\theta}}{6\theta^4}, \quad y \geq 0$$

Estimate: $\theta_e = \frac{1}{4n}\sum_{i=1}^{n} y_i$ ← Number

Estimator: $\hat{\theta} = \frac{1}{4n}\sum_{i=1}^{n} Y_i$ ← Random Variable

Check if $\hat{\theta}$ is an unbiased estimator:

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}\left(\frac{1}{4n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{4n}\sum_{i=1}^{n}\mathbb{E}(Y_i)$$

$$= \frac{1}{4n}\sum_{i=1}^{n} \mathbb{E}(Y) = \frac{1}{4n}\cdot n\,\mathbb{E}(Y) = \frac{1}{4}\mathbb{E}(Y).$$

Compute $\mathbb{E}(Y)$:

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y\cdot f(y;\theta)\,dy = \int_0^\infty y\cdot\frac{y^3 e^{-y/\theta}}{6\theta^4}\,dy$$

$$= \frac{1}{6\theta^4}\underbrace{\int_0^\infty y^4 e^{-y/\theta}\,dy}_{\star}$$

$$\star = \cancel{-\theta y^4 e^{-y/\theta}\Big|_0^\infty}\to 0 + \int_0^\infty 4\theta y^3 e^{-y/\theta}\,d\theta \qquad \begin{array}{ll} u=y^4 & dv = e^{-y/\theta}dy \\ du = 4y^3dy & v = -\theta e^{-y/\theta}\end{array}$$

$$= 4\theta\int_0^\infty y^3 e^{-y/\theta}\,dy \qquad \begin{array}{ll} u=y^3 & dv = e^{-y/\theta}dy \\ du = 3y^2dy & v = -\theta e^{-y/\theta}\end{array}$$

$$= 4\theta\left[\cancel{-\theta y^3 e^{-y/\theta}\Big|_0^\infty}\to 0 + \int_0^\infty 3\theta y^2 e^{-y/\theta}\,d\theta\right]$$

$$= 12\theta^2\int_0^\infty y^2 e^{-y/\theta}\,d\theta \qquad \begin{array}{ll} u=y^2 & dv = e^{-y/\theta}dy \\ du = 2y\,dy & v = -\theta e^{-y/\theta}\end{array}$$

$$= 12\theta^2\left[\cancel{-\theta y^2 e^{-y/\theta}\Big|_0^\infty}\to 0 + \int_0^\infty 2\theta y e^{-y/\theta}\,dy\right]$$

$$= 24\theta^3 \int_0^\infty y e^{-y/\theta}\,dy \qquad u = y \quad dv = e^{-y/\theta}dy$$
$$du = dy \quad v = -\theta e^{-y/\theta}$$

$$= 24\theta^3 \left[\underbrace{-y\theta e^{-y/\theta}\Big|_0^\infty}_{\to 0} + \int_0^\infty \theta e^{-y/\theta}\,dy \right]$$

$$= 24\theta^4 \int_0^\infty e^{-y/\theta}\,dy$$

$$= 24\theta^4 \left(-\theta e^{-y/\theta}\Big|_0^\infty \right)$$

$$= -24\theta^5 \left(e^{-y/\theta}\Big|_0^\infty \right)$$

$$= -24\theta^5 \left(0 - e^0 \right) = -24\theta^5(-1) = 24\theta^5$$

SO: $\star = 24\theta^5$

<u>Therefore</u>:

$$E(Y) = \frac{1}{6\theta^4} \underbrace{\int_0^\infty y^4 e^{-y/\theta}\,dy}_{\star} = \frac{1}{6\theta^4}(24\theta^5)$$
$$= 4\theta$$

<u>AND</u>:

$$E(\hat{\theta}) = \frac{1}{4}E(Y) = \frac{1}{4}\cdot 4\theta = \theta$$

<u>Conclusion</u>: $\hat{\theta} = \frac{1}{4n}\sum_{i=1}^{n} Y_i$ is an <u>unbiased</u> estimator for θ (Wow!)