Problem 1 *Let $P_1, P_2, \ldots, P_n$ be polygons no two of which are similar. Show that there are polygons $Q_1, Q_2, \ldots, Q_n$ with Q_i similar to P_i so that for no $i \neq j$ does Q_i contain a polygon congruent to Q_j.*

Proof. Let Q_i be the scaling of P_i that has area 1. Then if Q_i contains a copy of Q_j, since they have the same area, that copy must be all of Q_i. Thus, Q_i must be congruent to Q_j and hence P_i similar to P_j, which is a contradiction if $i \neq j$.

Problem 2 *Let $P(X)$ be a polynomial with real coefficients such that $P(X^n) \mid P(X^{n+1})$, for all $n \in \mathbb{N}$. Prove that $P(X) = cX^k$, for some real constant c and $k \in \mathbb{N}$.*

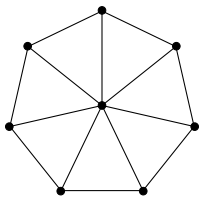
Proof. Suppose for sake of contradiction that $P(X)$ is not of this form. Then $P(X)$ must have some non-zero complex root ζ. Since $P(X^{n+1})$ has root $\zeta^{1/(n+1)}$ (in fact every $(n+1)st$ root of ζ is a root of $P(X^{n+1})$). Therefore, $\zeta^{1/(n+1)}$ must be a root of $P(X^n)$, so $\zeta^{n/(n+1)}$ must be a root of $P(X)$. But if we consider the $n+1$ different $(n+1)st$ roots of ζ, this means that P must have at least $n+1$ distinct roots, and hence have degree at least $n+1$. But this cannot hold for all n.

Thus, $P(X) = cX^k$ are the only possible solutions (and it is easy to verify that they work).

Alternative Solution: Suppose that $P(X)$ has leading term cX^k and suppose that $n > k$. Then $P(X^n)$ equals cX^{kn} plus terms of degree at most $kn - k - 1$ and $P(X^{n+1})$ equals cX^{kn+k} plus terms of degree less than kn. Suppose that the ratio of these polynomials is $Q(X)$. Clearly $\deg(Q) = \deg(P(X^{n+1})) - \deg(P(X^n)) = k$. We also have that $Q(X)P(X^n) = cX^{nk}Q(X)$ plus a polynomial of degree less than nk. Since this must equal $P(X^{n+1})$ which is cX^{kn+k} plus a polynomial of degree less than nk, we have $cX^{kn}(Q(X) - X^k)$ is a polynomial of degree less than nk, which implies that $Q(X) = X^k$. But then if the minimum degree term of $P(X) = aX^m$, the minimum degree term of $Q(X)P(X^n)$ is aX^{mn+k} and the minimum degree term of $P(X^{n+1})$ is aX^{mn+m}. Thus, we must have $m = k$ and thus the only non-zero term of $P(X)$ is cX^k.

Problem 3 *Consider a collection of $2n$ points in the plane, no three of which are collinear, and some set of line segments between them. We say that a subset of these line segments is a* pairing *if every one of these points is the endpoint of exactly one of the chosen line segments. Show that for every $k \geq 1$, there exists such an arrangement of points and line segments (for some value of $n \geq 1$) such that there are exactly k distinct pairings.*

Proof. For k is odd, we have the wheel graph where the internal vertex has degree k:



The center node can connect to any one of the k spoke nodes and then there is a unique way to extend the pairing, by letting each node around the cycle connect to the one next to it.

Also note that the square graph has exactly 2 pairings. For general n, we may write $n = k \cdot 2^m$ for some odd integer k and $m \geq 0$. We can then use the k wheel and m disconnected squares.

Problem 4 *Let $a_1, a_2, a_3, \ldots$ be the sequence $2, 1, 1, 2, \ldots$ so that a_i is 1 or 2 for each i and so that the decimal number $a_n a_{n-1} \cdots a_1$ is divisible by 2^n for each $n \geq 1$. Show that the decimal $0.a_1 a_2 a_3 \ldots$ is irrational.*

Proof. Suppose for sake of contradiction that $0.a_1 a_2 a_3 \ldots$ is rational. This means that this sequence eventually repeats. In particular, there must be integers $n > m$ so that for all $k \geq 0$ we have $a_{n+k} = a_{m+k}$.

Consider the decimals $B_n = 0.a_n a_{n-1} \ldots a_1 = [a_n a_{n-1} \cdots a_1]/10^n = C_n/5^n$ for some integer C_n (because $[a_n a_{n-1} \cdots a_1]$ is divisible by 2^n).

Notice that

$$B_{n+k} = 0.a_{n+k} a_{n+k-1} \ldots a_{n+1} + B_n/10^k,$$

and

$$B_{m+k} = 0.a_{m+k} a_{m+k-1} \ldots a_{m+1} + B_m/10^k.$$

So

$$B_{n+k} - B_{m+k} = (B_n - B_m)/10^k.$$

However, $B_{n+k} - B_{m+k} = (C_{n+k} - 5^{n-m} C_{m+k})/5^{n+k}$ is a rational number with no powers of 2 in the denominator, while $(B_n - B_m)/10^k$ has powers of 2 in the denominator if k is more than the number of factors of 2 in the numerator of $B_n - B_m$. This is a contradiction so long as $B_n - B_m \neq 0$, which is the case since B_n and B_m are decimals with different numbers of non-zero digits.