68thANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 26th, 2025
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number at the top of **each page of the contest**. (Keep your ID number handy for Part II.)
2. Calculators or other electronic devices may **not** be used.
3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, or put a clear X, or a checkmark; just make sure your selection is clear.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on May 9th.

Name: ID Number:

Problem 1 *Let x_1, x_2, x_3 be three distinct roots of the equation $x^3 - 4x + 1 = 0$. What is the value of $x_1^3 + x_2^3 + x_3^3$?*
◯ *(A)* -3
◯ *(B)* -1
◯ *(C)* 0
◯ *(D)* $\sqrt{2}$
◯ *(E)* 4

Problem 2 *What is the coefficient of x in the expansion of the expression $(x^3 + \frac{1}{x^2})^7$?*
◯ *(A)* 7
◯ *(B)* 15
◯ *(C)* 27
◯ *(D)* 32
◯ *(E)* 35

Problem 3 *A sphere is inscribed in a cube, which is inscribed in a larger sphere. What is the ratio of the surface area of the larger sphere to that of the smaller one?*
◯ *(A)* $\sqrt{3}$
◯ *(B)* 2
◯ *(C)* 3
◯ *(D)* π
◯ *(E)* 4

Problem 4 *For how many integer values of x is $x^4 - 2026x^2 + 2025$ negative?*
◯ *(A) 43*
◯ *(B) 44*
◯ *(C) 56*
◯ *(D) 82*
◯ *(E) 86*

Problem 5 *Consider an isosceles trapezoid with parallel segments of length $2\sqrt{3}-1$ and $2\sqrt{3}+1$ (the non-parallel segments have equal length, but are unspecified). Given that the area of the trapezoid is 6, find the smallest interior angle.*
◯ *(A)* $\pi/12$
◯ *(B)* $\pi/6$
◯ *(C)* $\pi/3$
◯ *(D)* $\pi/2$
◯ *(E) None of the above*

Problem 6 *How many pairs of non-negative integers (a, b) with $0 \leq a \leq b \leq 100$ have the arithmetic mean of a and b exactly one more than their geometric mean?*
◯ *(A)* 7
◯ *(B)* 8
◯ *(C)* 9
◯ *(D)* 10
◯ *(E)* 11

Problem 7 *A square casts a shadow (from a source infinitely far away) onto a flat plane. What is the shape of the shadow. Please give the most specific answer that is always true.*
◯ *(A) A square*
◯ *(B) A rectangle*
◯ *(C) A rhombus*
◯ *(D) A parallelogram*
◯ *(E) A quadrilateral*

Problem 8 *Let $n \geq 1$ be an integer. What is the probability that three numbers a, b, c drawn at random from $\{n-1, n, \ldots, 2n-1, 2n\}$ with replacement are not the side lengths of a (possibly degenerate) triangle?*
◯ *(A)* $\frac{1}{n!}$
◯ *(B)* $\frac{n}{(n+2)^3}$
◯ *(C)* $\frac{3}{(n+2)^3}$
◯ *(D)* $\frac{12}{(n+2)^3}$
◯ *(E)* $\frac{n}{n^3+1}$

Problem 9 *Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is $\sum_{n=1}^{2024} \frac{1}{S_n+S_{n-1}}$?*
◯ *(A)* $\sqrt{45}$
◯ *(B)* 44
◯ *(C)* $\sqrt{2024}$
◯ *(D)* 45
◯ *(E)* $2025 - \sqrt{2024}$

Problem 10 *Let x be a positive real number such that $x^2 + \frac{1}{x^2} = 3$. What is the value of $x^3 + \frac{1}{x^3}$?*
◯ *(A)* $\sqrt[3]{5}$
◯ *(B)* $\sqrt{10}$
◯ *(C)* $2\sqrt{5}$
◯ *(D)* 5
◯ *(E)* $3\sqrt{3}$

Problem 11 *Define a 2-progression to be a finite increasing sequence of positive integers of length at least 2 such that each integer in the sequence is 2 more than the integer before it. How many 2-progressions sum to 2025?*

◯ *(A)* 5
◯ *(B)* 6
◯ *(C)* 7
◯ *(D)* 8
◯ *(E)* 9

Problem 12 *A fair six-sided die is rolled n times. What is the probability that the product of the rolls is not a multiple of 4?*

◯ *(A)* $\frac{3n+1}{6^n}$
◯ *(B)* $\frac{2}{3^n}$
◯ *(C)* $\frac{2^n+3^n}{6^n}$
◯ *(D)* $\frac{2n+3}{3\cdot 2^n}$
◯ *(E)* $\frac{4}{n^3}$

Problem 13 *Suppose a triangle with integer side lengths has perimeter 20. What is the maximum possible area it can achieve?*

◯ *(A)* $5\sqrt{6}$
◯ *(B)* $4\sqrt{15}$
◯ *(C)* $8\sqrt{5}$
◯ *(D)* $6\sqrt{10}$
◯ *(E)* $10\sqrt{6}$

Problem 14 *What is the largest $c > 0$ such the inequality $x^3 + \frac{1}{\sqrt{2}} \geq cx$ holds for every $x > 0$?*

◯ *(A)* 1
◯ *(B)* $\frac{4}{3}$
◯ *(C)* $\sqrt{2}$
◯ *(D)* $\frac{3}{2}$
◯ *(E)* $\sqrt{3}$

Problem 15 *How many pairs of positive integers (n, m) are there for which $n^{m^2} = m^n$?*

◯ *(A)* 0
◯ *(B)* 1
◯ *(C)* 2
◯ *(D)* 3
◯ *(E) Infinitely Many*

Problem 16 *What is the smallest value of n so that if you roll n fair, eight sided dice there is a probability of at least 50% that some number more than 4 appears at least twice.*
◯ *(A)* 3
◯ *(B)* 4
◯ *(C)* 5
◯ *(D)* 6
◯ *(E)* 7

Problem 17 *Let a, b, c be positive integers for which all three equations*

$$\begin{aligned} x^2 - 2ax + b &= 0 \\ x^2 - 2bx + c &= 0 \\ x^2 - 2cx + a &= 0 \end{aligned}$$

have at least one integer solution. How many such triples (a, b, c) exist?
◯ *(A)* 0
◯ *(B)* 1
◯ *(C)* 3
◯ *(D)* 27
◯ *(E) Infinitely many*

Problem 18 *Let $a(k)$ be the largest odd number dividing k. Find the largest possible real number b such that for all positive integers $n \geq 2$,*

$$\frac{1}{2^n} \sum_{k=1}^{2^n} \frac{a(k)}{k} \geq b \, .$$

◯ *(A)* $1/4$
◯ *(B)* $1/2$
◯ *(C)* $2/3$
◯ *(D)* $3/4$
◯ *(E)* 1

Problem 19 *For which of the following values of x does 2025* not *divide $45x + x^2 + 77x^3$:*
◯ *(A) 79830*
◯ *(B) 105525*
◯ *(C) 148530*
◯ *(D) 149490*
◯ *(E) 444420*

Problem 20 *Which is the median of the numbers below?*
◯ *(A)* 220^5
◯ *(B)* 5^{20}
◯ *(C)* $22^{2\cdot 5}$
◯ *(D)* $2^{\sqrt{2025}}$
◯ *(E)* $20^{2\cdot 5}$

Problem 21 *Let a, b, c be three positive integers larger than 1 such that $a \mid b^{10}$, $b \mid c^{10}$, $c \mid a^{10}$. Let $n(a,b,c) = \min\{k \in \mathbb{N} \,:\, abc \mid (a+b+c)^k\}$. What is the maximal possible value of $n(a,b,c)$?*
◯ *(A)* 10
◯ *(B)* 100
◯ *(C)* 111
◯ *(D)* 1000
◯ *(E) $n(a,b,c)$ could be arbitrarily large*

Problem 22 *Consider a bag with 11 marbles of two different colors. The number of marbles of each color is fixed, but unknown. Draw marbles one at a time from a bag without replacement until you can determine with certainty which color is most common. How many possible distinct sequences of colors could you form in this way?*
◯ *(A) 84*
◯ *(B) 210*
◯ *(C) 462*
◯ *(D) 792*
◯ *(E) 924*

Problem 23 *Let $n \geq 1$ be an integer. What is the number of functions $f : \{1, 2, \ldots, 4n\} \to \{1, 2, \ldots, 4n\}$ such that $f(f(k)) = 4n + 1 - k$, for every $k \in \{1, 2, \ldots, 4n\}$?*
◯ *(A)* $2^n(2n-1)$
◯ *(B)* $\frac{(2n)!}{2^n n!}$
◯ *(C)* $2^n n!$
◯ *(D)* $\frac{(2n)!}{n!}$
◯ *(E)* $\frac{(4n)!}{(2n)!}$

Problem 24 *Three spheres are placed in three dimensional spaces. Exactly n planes are tangent to all three spheres (where n is possibly infinite). How many possible values can n have?*
◯ *(A)* 2
◯ *(B)* 5
◯ *(C)* 9
◯ *(D)* 10
◯ *(E) Infinitely many*

Problem 25 *How many polynomials p of degree at most 6 are there so that $p(n)$ is either 0 or 1 for $n = 1, 2, 3, \ldots, 8$?*

◯ *(A)* 0
◯ *(B)* 2
◯ *(C)* 4
◯ *(D)* 8
◯ *(E)* 16