Problem 1 *For real numbers x let $[x]$ denote the largest integer less than or equal to x, and let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ denote the set of non-negative integers. For a positive real number, a, define the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ given by $f(n) = [na]$. Show that if f is a bijection, then $a = 1$.*

If $a < 1$, then $f(0) = [0] = 0$ and $f(1) = [a] = 0$ and so f is not bijective.

If $a > 1$, let $b = a - 1 > 0$. Let n be the smallest integer so that $nb \geq 1$. Then $f(n-1) = [(n-1) + (n-1)b] = n - 1$ and $f(n) = [n + nb] \geq [n + 1] \geq n + 1$. Since f is an increasing function, there cannot be any value of m so that $f(m) = n$, and thus f is not bijective.

Problem 2 *We call a number $n \in \mathbb{N}$* square-free *if it is not divisible by any perfect square bigger than 1. We call n* perfect *if it has the property that*

$$\sum_k d_k = n \ ,$$

i where $d_1, \ldots, d_k$ are all the divisors of n that are smaller than n. Find all square-free perfect numbers.

The only such n is 6.

Suppose for sake of contradiction that such an n exists and consider its prime factorization. Since n is square-free, it cannot be divisible by the square of any prime, so it is the product of distinct primes. Namely, $n = p_1 p_2 \cdots p_k$, for some primes $p_1 < p_2 < \ldots < p_k$. Now the sum of all factors of n is $(p_1 + 1)(p_2 + 1) \cdots (p_k + 1)$, and if n is perfect, this must be $2n$ (since the sum of all divisors other than n add to n).

Since n is square-free, it is not divisible by 4, and thus $2n$ has at most 2 factors of 2. Thus, at most two of the (p_i+1) terms can be even. However, $p+1$ is even for every prime other than 2. Therefore n is either a single odd prime p, or 2 times the product of at most two other odd primes.

In the first case $n = p$, we must have $2n = p + 1$, which is impossible.

In the second case $n = 2p$ or $n = 2pq$.

If $n = 2p$, $2n = 4p = 3(p + 1)$. This implies that 3 divides p, so p must equal 3 and $n = 6$, which works.

If $n = 2pq$, then $2n = 4pq = 3(p + 1)(q + 1)$. So 3 must divide pq and so WLOG, $p = 3$. Then $12q = 12(q + 1)$, which is impossible.

Thus, the only solution is $n = 6$.

Problem 3 *Let A, B and C be three points in the plane, not all on a line. Show that for a positive integer n it is possible to draw n lines through A, n lines through B and n lines through C so that there are at least $n^2/2$ points that are at the intersection of three of these lines.*

Let the triangle ABC have sides $S_A = BC$, $S_B = AC$ and $S_C = CA$. Note that for any positive deal number α, there is a line through A containing all of the points x inside ABC so that $\frac{d(x,S_B)}{d(x,S_C)} = \alpha$. Similarly, for β and γ we can create lines through B and C that contain all of the points inside the triangle with $\frac{d(x,S_C)}{d(x,S_A)} = \beta$ and $\frac{d(x,S_A)}{d(x,S_B)} = \gamma$.

We now define our lines using this construction. In particular, we define our lines through A using $\alpha = 2^1, 2^2, \ldots, 2^n$, we define our lines through B with $\beta = 2^1, 2^2, \ldots, 2^n$ and define our lines through C with $\gamma = 2^{-n-1}, 2^{-n-2}, \ldots, 2^{-2n}$.

We now have to count intersections. Note that any of our lines through A and lines through B intersect at a point inside the triangle with

$$\frac{d(x,S_A)}{d(x,S_B)} = \left(\frac{d(x,S_B)}{d(x,S_C)}\right)^{-1}\left(\frac{d(x,S_C)}{d(x,S_A)}\right)^{-1} = \frac{1}{\alpha\beta}.$$

Therefore, this point of intersection will be on one of our lines through C if and only if $1/(\alpha\beta)$ is one of our selected γ's. This will happen if $\alpha = 2^a, \beta = 2^b$ and $a + b \geq n + 1$. Note that for each $a = 1, \ldots, n$, this leaves a many possible $b's$. Thus, the total number of 3-way intersections is at least $1 + 2 + \ldots + n = n(n+1)/2 > n^2/2$.

Note: It is actually possible to get roughly $3n^2/4$ intersections.

Problem 4 *Let $\mathcal{A}$ be a family of sets of finite size such that $|\mathcal{A}| = 64$ and so that $A \cup B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$. Prove that there exists an element x contained in at least 16 sets in $\mathcal{A}$.*

Let X be a set of minimal size so that every non-empty set in $\mathcal{A}$ has a non-empty intersection with X. We split into cases.

If $|X| \leq 4$, then all but one element of $\mathcal{A}$ will contain at least one of the at most 4 elements of X. Therefore, at least one element of X will be in at least a quarter of the non-empty elements of $\mathcal{A}$, and thus be in at least 16 sets.

If $|X| \geq 5$, we first make use of the minimality of X. In particular, for any $x \in X$, there must be some $A \in \mathcal{A}$ so that $A \cap X = \{x\}$. This is because otherwise we could remove x from X and it would still intersect every non-empty set of A. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $A_i \in \mathcal{A}$ has $A_i \cap X = \{x_i\}$. By taking the union of $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$, we can find an $A \in \mathcal{A}$ with $A \cap X = \{x_{i_1}, \ldots, x_{i_m}\}$. In particular, for every subset S of X, there is an $A \in \mathcal{A}$ with $A \cap X = S$. But for a given $x \in X$, there are $2^{|X|-1}$ many such subsets that contain x, giving us at least $2^{|X|-1} \geq 2^{5-1} = 16$ elements of $\mathcal{A}$ that contain x.