67th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 20th, 2024
on the campus of the
University of California, San Diego

PART II
4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name _____
First Last

School _____

Contest ID number _____

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number on **each page of the contest**.
2. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.
3. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.
4. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Banquet on May 6th.

Name: ID Number:

Problem 1 *For real numbers x let $[x]$ denote the largest integer less than or equal to x, and let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ denote the set of non-negative integers. For a positive real number, a, define the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ given by $f(n) = [na]$. Show that if f is a bijection, then $a = 1$.*

Problem 2 *We call a number $n \in \mathbb{N}$* square-free *if it is not divisible by any perfect square bigger than 1. We call n* perfect *if it has the property that*

$$\sum_k d_k = n \ ,$$

where $d_1, \ldots, d_k$ are all the divisors of n that are smaller than n. Find all square-free perfect numbers.

Problem 3 *Let A, B and C be three points in the plane, not all on a line. Show that for a positive integer n it is possible to draw n lines through A, n lines through B and n lines through C so that there are at least $n^2/2$ points that are at the intersection of three of these lines.*

Problem 4 *Let $\mathcal{A}$ be a family of sets of finite size such that $|\mathcal{A}| = 64$ and so that $A \cup B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$. Prove that there exists an element x contained in at least 16 sets in $\mathcal{A}$.*