67th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 20th, 2024
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number at the top of **each page of the contest**. (Keep your ID number handy for Part II.)
2. Calculators or other electronic devices may **not** be used.
3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, or put a clear X, or a checkmark; just make sure your selection is clear.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on May 6th.

Name: ID Number:

Problem 1 *A recipe calls for* 2 *cups flour and* 1 *cup of lowfat* (1% *milkfat) milk. Jeremy has already added* 1 *cup of flour and half a cup of whole* (4% *milkfat) milk. He plans to fix this by adding skim milk* (0% *milkfat) and an appropriate amount of flour in order to make some multiple of the original recipe in the correct proportions. What is the minimum number of cups of flour that he needs to add?*

◯ *(A)* 0
◯ *(B)* 1
◯ *(C)* 2
◯ *(D)* 3
◯ *(E)* 4

Problem 2 *The circumference of a circle is equal to the perimeter of a square. What is the ratio of the area of the circle to the area of the square.*

◯ *(A)* 1
◯ *(B)* $\frac{\pi}{3}$
◯ *(C)* $\frac{2}{\sqrt{\pi}}$
◯ *(D)* $\frac{4}{\pi}$
◯ *(E)* $\sqrt{2}$

Problem 3 *Consider a point* p *on a sphere of radius* 1*. Let* S *be the set of points on the surface of the sphere whose straight line distance from* p *is equal to* d*. If* S *forms a circle of radius* $3/5$*, what is* d*?*

◯ *(A)* $1/2$
◯ *(B)* $\sqrt{2/5}$
◯ *(C)* $\sqrt{1/2}$
◯ *(D)* $\sqrt{3/5}$
◯ *(E) None of the above*

Problem 4 *Simplify the expression*

$$(\sqrt{2}+\sqrt{3}+\sqrt{5})(-\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})$$

◯ *(A)* $\sqrt{30}$
◯ *(B)* $\sqrt{6}+\sqrt{10}+\sqrt{15}$
◯ *(C)* $4\sqrt{6}$
◯ *(D)* 12
◯ *(E)* 24

Problem 5 *A right triangle has area* 7 *and perimeter* 14*. What is the length of the hypothenuse?*
◯ *(A)* $4+2\sqrt{2}$
◯ *(B)* 6
◯ *(C)* $2\sqrt{7}+1$
◯ *(D)* 7
◯ *(E)* 8

Problem 6 *What is the largest possible area of a rectangle whose diagonal has length 1?*
◯ *(A)* $1/3$
◯ *(B)* $1/2$
◯ *(C)* $2/3$
◯ *(D)* $1/\sqrt{2}$
◯ *(E)* 1

Problem 7 *Hanna's exercise routine involves running for* 2 *minutes, rolling a six-sided die and resting for a number of minutes given by the result before repeating this process. If she continues like this for a long period of time, roughly what fraction of that time will be spent running?*
◯ *(A)* $2/9$
◯ *(B)* $1/4$
◯ *(C)* $4/11$
◯ *(D)* $341/840$
◯ *(E)* $4/7$

Problem 8 *How many integers can be chosen in* $\{1,2,\ldots,20\}$ *so that no two of the chosen numbers have a product that is a perfect square?*
◯ *(A)* 9
◯ *(B)* 10
◯ *(C)* 11
◯ *(D)* 12
◯ *(E)* 13

Problem 9 *Let x_1, x_2, x_3, x_4 be real numbers such that $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. What is the largest possible value that $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$ can have?*

◯ *(A)* -1
◯ *(B)* $-1/2$
◯ *(C)* 0
◯ *(D)* 1
◯ *(E) None of the above*

Problem 10 *Students $1, 2, \ldots, 2024$ take classes in college, and the college offers 2024 classes $C_1, C_2, \ldots, C_{2024}$. Student i take classes C_i, C_{i+1} and C_{i+2} where we define $C_{2025} = C_1$ and $C_{2026} = C_2$. A schedule is an assignment of times to classes (with each class assigned only one time) so that no student has to take two classes at the same time. The minimum number of times in a schedule is*

◯ *(A)* 2
◯ *(B)* 3
◯ *(C)* 4
◯ *(D)* 5
◯ *(E) None of the above*

Problem 11 *The largest number of subsets of $\{1, 2, 3, 4, 5, 6\}$ which can be chosen so any two of the sets have non-empty intersection is:*

◯ *(A)* 8
◯ *(B)* 16
◯ *(C)* 32
◯ *(D)* 40
◯ *(E) None of the above*

Problem 12 *What is the smallest $C > 0$ such that $\sqrt{a} + 2\sqrt{b} \leq C\sqrt{a+b}$, for all $a > 0$ and $b > 0$.*

◯ *(A)* 2
◯ *(B)* $\sqrt{5}$
◯ *(C)* $\frac{5}{2}$
◯ *(D)* $2\sqrt{2}$
◯ *(E)* 3

Problem 13 *The plane is tiled with congruent triangles. That is, congruent triangles are placed in the plane so that every point is contained in some triangle and so that the triangles only overlap on their edges. What is the greatest number of these triangles that can meet at a single vertex?*
◯ *(A)* 6
◯ *(B)* 8
◯ *(C)* 10
◯ *(D)* 12
◯ *(E) There is no greatest number*

Problem 14 *The largest size of a subset* S *of* $\{1, 2, 3, \ldots, 13\}$ *such that all differences* $a - b$ *with* $a, b \in S$ *and* $a \neq b$ *are distinct is*
◯ *(A)* 3
◯ *(B)* 4
◯ *(C)* 5
◯ *(D)* 6
◯ *(E)* 7

Problem 15 *A bunny is located on the number line starting at* 2*. From a number* n *it will jump to* $n + p$ *where* p *is the largest prime factor of* n*. What number does it end up on after* 200 *jumps?*
◯ *(A)* 10298
◯ *(B)* 10299
◯ *(C)* 10300
◯ *(D)* 10301
◯ *(E)* 10302

Problem 16 *A fair six-sided die is rolled* n *times. What is the probability that the product of the rolls is a power of* 4*? (Note that* 1 *is a power of* 4*.)*
◯ *(A)* $\frac{2^n+1}{6^n}$
◯ *(B)* $\frac{3^n+1}{2 \cdot 6^n}$
◯ *(C)* $\frac{1}{2^n}$
◯ *(D)* $\frac{2^n+3^n}{6^n}$
◯ *(E)* $\frac{1}{n^2}$

Problem 17 *How many pairs of integers (a, b) are there such that $x^3 + ax^2 + 2024x + b$ has distinct prime roots.*
◯ *(A)* 0
◯ *(B)* 1
◯ *(C)* 2
◯ *(D)* 3
◯ *(E) More than* 3

Problem 18 *Let P be a convex polygon in the plane with four sides. Suppose all sides have length 1, except one side has length $x \geq 1$. Then the maximum possible area of P is*
◯ *(A)* 1
◯ *(B)* $3\sqrt{3}/4$
◯ *(C)* $\sqrt{2}$
◯ *(D)* $\sqrt{3}$
◯ *(E) None of the above*

Problem 19 *If a, b, c are the roots of the polynomial $x^3 - 3x + 1$, then ab, ac, bc are the roots of the polynomial:*
◯ *(A)* $x^3 - x^2 - x$
◯ *(B)* $x^3 + x - 3$
◯ *(C)* $x^3 + 3x - 1$
◯ *(D)* $x^3 + 3x^2 - 1$
◯ *(E)* $3x^3 - x^2 + 1$

Problem 20 *Arrange the positive integers 1 through 11 in order around a circle. What is the sum of all products of 9 consecutive numbers around the circle?*
◯ *(A)* $10!$
◯ *(B)* $10 \cdot (10!)$
◯ *(C)* $11!$
◯ *(D)* $2 \cdot (11!)$
◯ *(E)* $12!$

Problem 21 *How many (possibly empty) subsets of $\{1, 2, \ldots, 8\}$ are there such that no two elements in the subset differ by exactly 2?*
◯ *(A)* 64
◯ *(B)* 65
◯ *(C)* 81
◯ *(D)* 82
◯ *(E)* 100

Problem 22 *Let L be an ordered list of 2024 distinct integers from 1 to 2024. Let $\mathsf{Rotate}(k)$ be the operation which rotates the first k elements of a list—i.e., the first element is placed at the second position, the second is placed in the third position, and so on until the kth element is placed in the first position. How many ways are there to put L in sorted order using a sequence of rotations $\mathsf{Rotate}(a_1), \mathsf{Rotate}(a_2), \ldots, \mathsf{Rotate}(a_m)$ where $a_1 \leq a_2 \leq \cdots \leq a_m$ without ever allowing the list to be in the same order more than once?*
◯ *(A)* 0
◯ *(B)* 1
◯ *(C)* 2
◯ *(D)* 2024
◯ *(E) Depends on the contents of L*

Problem 23 *Elijah walks one mile due North, then one mile due East, then one mile due South and finally one mile due West, arriving back at where he started. Assuming that he started north of the equator, what is the furthest from the North pole that he can be in miles?*
◯ *(A)* $(-1+\sqrt{1+2\pi})/2$
◯ *(B)* 1
◯ *(C)* $1+1/(2\pi)$
◯ *(D)* $\pi/(\pi-1)$
◯ *(E)* $(1+\sqrt{1+2/\pi})/2$

Problem 24 *Let N be the smallest integer so that $d(N)=2024$, where $d(N)$ denotes the number of divisors of N. What are the last two digits of N?*
◯ *(A)* 00
◯ *(B)* 20
◯ *(C)* 40
◯ *(D)* 60
◯ *(E)* 80

Problem 25 *Say that a line L in the coordinate plane is far, if no point on L has distance less than $1/10$ from any point with integer coordinates. How many possible slopes could a far line have?*
◯ *(A)* 24
◯ *(B)* 26
◯ *(C)* 28
◯ *(D)* 32
◯ *(E) Infinitely many*