Supplements to the Exercises in Chapters 1-7 of Walter Rudin's *Principles of Mathematical Analysis, Third Edition*

by George M. Bergman

This packet contains both additional exercises relating to the material in Chapters 1-7 of Rudin, and information on Rudin's exercises for those chapters. For each exercise of either type, I give a title (an idea borrowed from Kelley's *General Topology*), an estimate of its difficulty, notes on its dependence on other exercises if any, and sometimes further comments or hints.

Numbering. I have given numbers to the sections in each chapter of Rudin, in general taking each of his capitalized headings to begin a new numbered section, though in a small number of cases I have inserted one or two additional section-divisions between Rudin's headings. My exercises are referred to by boldfaced symbols showing the chapter and section, followed by a colon and an exercise-number; e.g., under section 1.4 you will find Exercises **1.4:1**, **1.4:2**, etc.. Rudin puts his exercises at the ends of the chapters; in these notes I abbreviate ''Chapter M, Rudin's Exercise N'' to ***M*:R*N***. However, I list both my exercises and his under the relevant section.

It could be argued that by listing Rudin's exercises by section I am effectively telling the student where to look for the material to be used in solving the exercise, which the student should really do for his or her self. However, I think that the advantage of this work of classification, in showing student and instructor which exercises are appropriate to attempt or to assign after a given section has been covered, outweighs that disadvantage. Similarly, I hope that the clarifications and comments I make concerning many of Rudin's exercises will serve more to prevent wasted time than to lessen the challenge of the exercises.

Difficulty-codes. My estimate of the difficulty of each exercise is shown by a code **d**: 1 to **d**: 5. Codes **d**: 1 to **d**: 3 indicate exercises that it would be appropriate to assign in a non-honors class as ''easier'', ''typical'', and ''more difficult'' problems; **d**: 2 to **d**: 4 would have the same roles in an honors course, while **d**: 5 indicates the sort of exercise that might be used as an extra-credit ''challenge problem'' in an honors course. If an exercise consists of several parts of notably different difficulties, I may write something like **d**: 2, 2, 4 to indicate that parts (a) and (b) have difficulty 2, while part (c) has difficulty 4. However, you shouldn't put too much faith in my estimates – I have only used a small fraction of these exercises in teaching, and in other cases my guesses as to difficulty are very uncertain. (Even my sense of what level of difficulty should get a given code has probably been inconsistent. I am inclined to rate a problem that looks straightforward to me **d**: 1; but then I may remember students coming to office hours for hints on a problem that looked similarly straightforward, and change that to **d**: 2.)

The *difficulty* of an exercise is not the same as the *amount of work* it involves – a long series of straightforward manipulations can have a low level of difficulty, but involve a lot of work. I discovered how to quantify the latter some years ago, in an unfortunate semester when I had to do my own grading for the basic graduate algebra course. Before grading each exercise, I listed the steps I would look for if the student gave the expected proof, and assigned each step one point (with particularly simple or complicated steps given ½ or 1½ points). Now for years, I had asked students to turn in weekly feedback on the time their study and homework for the course took them; but my success in giving assignments that kept the average time in the appropriate range (about 13 hours per week on top of the 3 hours in class) had been erratic; the time often ended up far too high. That Semester, I found empirically that a 25-point assignment regular kept the time quite close to the desired value.

I would like to similarly assign point-values to each exercise here, from which it should be possible to similarly calibrate assignments. But I don't have the time to do this at present.

Dependencies. After the title and difficulty-code, I note in some cases that the exercise depends on some other exercise, writing ''>'' to mean ''must be done after ...''.

Comments on Rudin's exercises. For some of Rudin's exercises I have given, after the above data, notes clarifying, motivating, or suggesting how to approach the problem. (These always refer the exercise listed immediately above the comment; if other exercises are mentioned, they are referred to by number.)

True/False questions. In most sections (starting with §1.2) the exercises I give begin with one numbered ''0'', and consisting of one or more True/False questions, with answers shown at the bottom of the *next* page. Students can use these to check whether they have correctly understood and absorbed the definitions, results, and examples in the section. No difficulty-codes are given for True/False questions. I tried to write them to check for the most elementary things that students typically get confused on, such as the difference between a statement and its converse, and order of quantification, and for the awareness of what Rudin's various counterexamples show. Hence these questions should, in theory, require no original thought; i.e., they should be ''**d**: 0'' relative to the classification described above. But occasionally, either I did not see a good way to give such a question, or I was, for better or worse, inspired with a question that tested the student's understanding of a result via a not-quite-trivial application of it.

URL: `http://www.math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.ps` *or* `.pdf`

Terminology and Notation. I have followed Rudin's notation and terminology very closely, e.g. using R for the field of real numbers, J for the set of positive integers, and ''at most countable'' to describe a set of cardinality $\leq \aleph_0$. But on a few points I have diverged from his notation: I distinguish between sequences (s_i) and sets $\{s_i\}$ rather than writing $\{s_i\}$ for both, and I use $\subseteq$ rather than $\subset$ for inclusion. I also occasionally use the symbols $\forall$ and $\exists$, since it seems worthwhile to familiarize the student with them.

Advice to the student. An exercise may only require you to use the *definitions* in the relevant section of Rudin, or it may require for its proof some *results* proved there, or an argument using the same *method of proof* as some result proved there. So in approaching each problem, first see whether the result becomes reasonably straightforward when all the relevant definitions are noted, and also ask yourself whether the statement you are to prove is related to the conclusion of any of the theorems in the section, and if so, whether that theorem can be applied as it stands, or whether a modification of the proof can give the result you need. (Occasionally, a result listed under a given section may require only material from earlier sections, but is placed there because it throws light on the ideas of the section.)

Unless the contrary is stated, solutions to homework problems are expected to contain proofs, even if the problems are not so worded. In particular, if a question asks whether something is (always) true, an affirmative answer requires a proof that it is always true, while a negative answer requires an example of a case where it fails. Likewise, if an exercise or part of an exercise says ''Show that this result fails if such-and-such condition is deleted'', what you must give is an example which satisfies all the hypotheses of the result except the deleted one, and for which the conclusion of the result fails. (I am not counting the true/false questions under ''homework problems'' in this remark, since they are not intended to be handed in; but when using these to check yourself on the material in a given section, you should *be able* to justify with a proof or counterexample every answer that is not simply a statement taken from the book.)

From time to time students in the class ask ''Can we use results from other courses in our homework?'' The answer is, in general, ''No.'' Rudin shows how the material of lower division calculus can be developed, essentially from scratch, in a rigorous fashion. Hence to call on material you have seen developed in the loose fashion of your earlier courses would defeat the purpose. Of course, there are certain compromises: As Rudin says, he will assume the basic properties of integers and rational numbers, so you have to do so too. Moreover, once one has developed rigorously the familiar laws of differentiation and integration (a minor aspect of the material of this course), the application of these is not essentially different from what you learned in calculus, so it is probably not essential to state explicitly in homework for later sections which of those laws you are using at every step. When in doubt on such matters, ask your instructor.

Unfinished business. I have a large list of notes on errata to Rudin, unclear points, proofs that could be done more nicely, etc., which I want to write up as a companion to this collection of exercises, when I have time. For an earlier version, see http://www.math.berkeley.edu/~gbergman/ug.hndts/Rudin_notes.ps.

As mentioned in the paragraph in small print on the preceding page, I would like to complement the ''difficulty ratings'' that I give each exercise with ''amount-of-work ratings''. I would also like to complement the dependency notes with reverse-dependency notes, marking exercises which later exercises depend on, since this can be relevant to an instructor's decision on which exercises to assign. This will require a bit of macro-writing, to insure that consistency is maintained as exercises are added and moved around, and hence change their numbering. On a much more minor matter, I want to rewrite the page-header macro so that the top of each page will show the section(s) of Rudin to which the material on the page applies.

I am grateful to Charles Pugh for giving me comments on an early draft of this packet. I would welcome further comments and corrections on any of this material.

George Bergman
Department of Mathematics
University of California
Berkeley, CA 94720-3840

gbergman@math.berkeley.edu

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Chapter 1. The Real and Complex Number Systems.

1.1. INTRODUCTION. (pp.1-3)

Relevant exercise in Rudin:

1:R2. *There is no rational square root of* 12. (**d**: 1)

Exercise not in Rudin:

1.1:1. *Motivating Rudin's algorithm for approximating* $\sqrt{2}$. (**d**: 1)

On p.2, Rudin pulls out of a hat a formula which, given a rational number p, produces another rational number q such that q^2 is closer to 2 than p^2 is. This exercise points to a way one could come up with that formula. It is not an exercise in the usual sense of testing one's grasp of the material in the section, but is given, rather, as an aid to students puzzled as to where Rudin could have gotten that formula. We will assume here familiar computational facts about the real numbers, including the existence of a real number $\sqrt{2}$, though Rudin does not formally introduce the real numbers till several sections later.

(a) By rationalizing denominators, get a non-fractional formula for $1/(\sqrt{2}+1)$. Deduce that if $x = \sqrt{2}+1$, then $x = (1/x)+2$.

(b) Suppose $y > 1$ is some approximation to $x = \sqrt{2}+1$. Give a brief reason why one should expect $(1/y)+2$ to be a *closer* approximation to x than y is. (I don't ask for a proof, because we are only seeking to motivate Rudin's computation, for which *he* gives an exact proof.)

(c) Now let $p > 0$ be an approximation to $\sqrt{2}$ (rather than to $\sqrt{2}+1$). Obtain from the result of (b) an expression $f(p)$ that should give a closer approximation to $\sqrt{2}$ than p is. (Note: To make the input p of your formula an approximation of $\sqrt{2}$, substitute $y = p+1$ in the expression discussed in (b); to make the output an approximation of $\sqrt{2}$, subtract 1.)

(d) If $p < \sqrt{2}$, will the value $f(p)$ found in part (c) be greater or less than $\sqrt{2}$? You will find the result different from what Rudin wants on p.2. There are various ways to correct this. One would be to use $f(f(p))$, but this would give a somewhat more complicated expression. A simpler way is to use $2/f(p)$. Show that this gives precisely $(2p+2)/(p+2)$, Rudin's formula (3).

(e) Why do you think Rudin begins formula (3) by expressing q as $p-(p^2-2)/(p+2)$?

1.1:2. *Another approach to the rational numbers near* $\sqrt{2}$. (**d**: 2)

Let sets A and B be the sets of rational numbers defined in the middle of p.2. We give below a quicker way to see that A has no largest and B no smallest member. Strictly speaking, this exercise belongs under §1.3, since one needs the tools in that section to do it. (Thus, it should not be assigned to be done before students have read §1.3, and students working it may assume that Q has the properties of an ordered field as described in that section.) But I am listing it here because it simplifies an argument Rudin gives on p.2.

Suppose A has a largest member p.

(a) Show that the rational number $p' = 2/p$ will be a smallest member of B.

(b) Show that $p' > p$.

(c) Let $q = (p+p')/2$, consider the two possibilities $q \in A$ and $q \in B$, and in each case obtain a contradiction. (Hint: Either the condition that p is the greatest element of A or that p' is the smallest element of B will be contradicted.)

This contradiction disproves the assumption that A had a largest element.

(d) Show that if B had a smallest element, then one could find a largest element of A. Deduce from the result of (c) that B cannot have a smallest element.

1.2. ORDERED SETS. (pp.3-5)

Relevant exercise in Rudin:

1:R4. *Lower bound ≤ upper bound.* (**d**: 1)

Exercises not in Rudin:

1.2:0. *Say whether each of the following statements is true or false.*

(a) If x and y are elements of an ordered set, then either $x \geq y$ or $y > x$.

(b) An ordered set is said to have the ''least upper bound property'' if the set has a least upper bound.

1.2:1. *Finite sets always have suprema.* (**d**: 1)

Let S be an ordered set (*not* assumed to have the least-upper-bound property).

(a) Show that every two-element subset $\{x, y\} \subseteq S$ has a supremum. (Hint: Use part (a) of Definition 1.5.)

(b) Deduce (using induction) that every finite subset of S has a supremum.

1.2:2. *If one set lies above another.* (**d**: 1)

Suppose S is a set with the least-upper-bound property and the greatest-lower-bound property, and suppose X and Y are nonempty subsets of S.

(a) If every element of X is $\leq$ every element of Y, show that $\sup X \leq \inf Y$.

(b) If every element of X is $<$ every element of Y, does it follow that $\sup X < \inf Y$? (Give a proof or a counterexample.)

1.2:3. *Least upper bounds of least upper bounds, etc.* (**d**: 2)

Let S be an ordered set with the least upper bound property, and let A_i $(i \in I)$ be a nonempty family of nonempty subsets of S. (This means that I is a nonempty index set, and for each $i \in I$, A_i is a nonempty subset of S.)

(a) Suppose each set A_i is bounded above, let $\alpha_i = \sup A_i$, and suppose further that $\{\alpha_i \mid i \in I\}$ is bounded above. Then show that $\bigcup_{i \in I} A_i$ is bounded above, and that $\sup(\bigcup_{i \in I} A_i) = \sup\{\alpha_i \mid i \in I\}$.

(b) On the other hand, suppose that either (i) not all of the sets A_i are bounded above, or (ii) they are all bounded above, but writing $\alpha_i = \sup A_i$ for each i, the set $\{\alpha_i \mid i \in I\}$ is unbounded above. Show in each of these cases that $\bigcup_{i \in I} A_i$ is unbounded above.

(c) Again suppose each set A_i is bounded above, with $\alpha_i = \sup A_i$. Show that $\bigcap_{i \in I} A_i$ is also bounded above. Must it be nonempty? If it is nonempty, what can be said about the relationship between $\sup(\bigcap_{i \in I} A_i)$ and the numbers α_i $(i \in I)$?

1.2:4. *Fixed points for increasing functions.* (**d**: 3)

Let S be a nonempty ordered set such that every nonempty subset $E \subseteq S$ has both a least upper bound and a greatest lower bound. (A closed interval $[a, b]$ in R is an example of such an S.) Suppose $f: S \to S$ is a monotonically increasing function; i.e., has the property that for all $x, y \in S$, $x \leq y \Rightarrow f(x) \leq f(y)$.

Show that there exists an $x \in S$ such that $f(x) = x$.

1.2:5. *If everything that is* $> \alpha$ *is* $\geq \beta$ ... (**d**: 2)

(a) Let S be an ordered set such that for any two elements $p < r$ in S, there is an element $q \in S$ with $p < q < r$. Suppose α and β are elements of S such that for every $x \in S$ with $x > \alpha$, one has $x \geq \beta$. Show that $\beta \leq \alpha$.

(b) Show by example that this does not remain true if we drop the assumption that whenever $p < r$ there is a q with $p < q < r$.

1.2:6. *L.u.b.'s can depend on where you take them.* (**d**: 3)

(a) Find subsets $E \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq Q$ such that E has a least upper bound in S_1, but does not have any least upper bound in S_2, yet does have a least upper bound in S_3.

(b) Prove that for any example with the properties described in (a) (not just the example you have given), the least upper bound of E in S_1 must be different from the least upper bound of E in S_3.

(c) Can there exist an example with the properties asked for in (a) such that $E = S_1$? (If your answer is yes, you must show this by giving such an example. If your answer is no, you must prove it impossible.)

1.2:7. *A simpler formula characterizing l.u.b.'s.* (**d**: 2)

Let S be an ordered set, E a subset of S, and x an element of S.

If one translates the statement "x is the least upper bound of E" directly into symbols, one gets

$$((\forall\, y \in E)\ x \geq y)\ \wedge\ ((\forall\, z \in S)\,((\forall\, y \in E)\ z \geq y) \Rightarrow z \geq x).$$

This leads one to wonder whether there is any simpler way to express this property.

Prove, in fact, that x is the least upper bound of E if and only if

$$(\forall\, y \in S)\ (y < x \Leftrightarrow ((\exists\, z \in E)\,(z > y))).$$

1.2:8. *Some explicit* sup*'s and* inf*'s.* (**d**: 2)

(a) Prove that $\inf\,\{x+y+z \mid x, y, z \in R,\ 0 < x < y < z\} = 0$.

(b) Determine the values of each of the following. If a set is not bounded on the appropriate side, answer "undefined". No proofs need be handed in; but of course you should reason out your answers to your own satisfaction.

$a = \inf\,\{x+y+z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$b = \inf\,\{x+y-z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$c = \inf\,\{x-y+z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$d = \sup\,\{x+y+z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$e = \sup\,\{x+y-2z \mid x, y, z \in R,\ 1 < x < y < z\}$.

1.3. FIELDS. (pp.5-8)

Relevant exercise in Rudin:

1:R3. *Prove Proposition 1.15.* (**d**: 1)

Exercise **1:R5** can also be done after reading this section, if one replaces "real numbers" by "elements of an ordered field with the least upper bound property".

Exercises not in Rudin:

1.3:0. *Say whether each of the following statements is true or false.*

(a) Z (the set of integers, under the usual operations) is a field.

(b) If F is a field and also an ordered set, then it is an ordered field.

(c) If x and y are elements of an ordered field, then $x^2+y^2 \geq 0$.

(d) In every ordered field, $-1 < 0$.

1.3:1. $\sup(\{s+y \mid s \in S\}) = (\sup S) + y$. (**d**: 1,1,2)

Let F be an ordered field.

(a) Suppose S is a subset of F and y an element of F, and let $T = \{s+y \mid s \in S\}$. Show that if S has a least upper bound, $\sup S$, then T also has a least upper bound, namely $(\sup S) + y$.

(b) Deduce from (a) that if x is a nonzero element of F and we let $S = \{nx \mid n \text{ is an integer}\}$, then S has no least upper bound.

(c) Deduce Theorem 1.20(a) from (b) above.

1.4. THE REAL FIELD. (pp.8-11)

Relevant exercises in Rudin:

1:R1. *Rational + irrational = irrational.* (**d**: 1)

("Irrational" means belonging to R but not to Q.)

Answers to True/False question **1.2:0.** (a) T. (b) F.

1:R5. $\inf A = -\sup(-A)$. (**d**: 1)

1:R6. *Rational exponentiation of positive real numbers.* (**d**: 3. >**1:R7**(*a-c*))

Part (*a*) is more easily done if the first display is changed from ``$(b^m)^{1/n} = (b^p)^{1/q}$'' to ``$(b^{1/n})^m = (b^{1/q})^p$'', and in the next line ``$(b^m)^{1/n}$'' is changed to ``$(b^{1/n})^m$''. Part (*d*) requires part (*c*) of the *next* exercise, so the parts of these two exercises should be done in the appropriate order.

1:R7. *Logarithms of positive real numbers.* (**d**: 3)

In part (*g*), ``is unique'' should be ``is the unique element satisfying the above equation''.

It is interesting to compare the statement proved in part (*a*) of this exercise with the archimedean property of the real numbers. That property says that if one takes a real number > 0 and adds it to itself enough times, one can get above any given real number. From that fact and part (*a*) of this exercise, we see that if one takes a real number > 1 and *multiplies* it by itself enough times, one can get above any given real number. One may call the former statement the ``additive archimedean property'', and this one the ``multiplicative archimedean property''.

Exercises not in Rudin:

1.4:0. *Say whether each of the following statements is true or false.*

(a) Every ordered field has the least-upper-bound property.

(b) For every real number x, $(x^2)^{1/2} = x$.

(c) $\sqrt{1/2} > 1/2$.

(d) If a subset E of the real numbers is bounded above, and $x = \sup E$, then $x \in E$.

(e) If a subset E of the real numbers has a largest element, x (i.e., if there exists an element $x \in E$ which is greater than every other element of E), then $x = \sup E$.

(f) If E is a subset of R, and s is a real number such that $s > x$ for all $x \in E$, then $s = \sup E$.

1.4:1. *Some explicit* sup*'s and* inf*'s.* (**d**: 2)

(a) Prove that $\inf \{x+y+z \mid x, y, z \in R,\ 0 < x < y < z\} = 0$.

(b) Determine the values of each of the following. If a set is not bounded on the appropriate side, answer ``undefined''. No proofs need be handed in; but of course you should reason out your answers to your own satisfaction.

$a = \inf \{x+y+z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$b = \inf \{x+y-z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$c = \inf \{x-y+z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$d = \sup \{x+y+z \mid x, y, z \in R,\ 1 < x < y < z\}$.

$e = \sup \{x+y-2z \mid x, y, z \in R,\ 1 < x < y < z\}$.

1.4:2. *Details on decimal expansions of real numbers.* (**d**: 3)

This exercise gives some of the details skipped over in Rudin's sketch of the decimal expansion of real numbers.

In parts (a) and (b) below, let x be a positive real number, and let $n_0, n_1, \ldots, n_k, \ldots$ be constructed as in Rudin's 1.22 (p.11).

(a) Prove that for all nonnegative integers k, one has $0 \le x - \Sigma_{i=0}^{k} n_i 10^{-i} < 10^{-k}$, and that for all positive integers k, $0 \le n_k < 10$.

We would like to conclude from the former inequality that x is the least upper bound of $\{\Sigma_{i=0}^{k} n_i 10^{-i} \mid k \ge 0\}$. However, in this course we want to prove our results, and to prove this, we need a fact about the numbers 10^{-k}. This is obtained in the next part:

(b) For any real number $c > 1$, show that $\{c^k \mid k \ge 0\}$ is not bounded above. (Hint: Write $c = 1+h$ and note that $c^n \ge 1+nh$. Then what?) Deduce that the greatest lower bound of $\{c^{-k} \mid k \ge 0\}$ is 0. Taking $c = 10$, show that this together with the result of (a) implies that the least upper bound of $\{\Sigma_{i=0}^{k} n_i 10^{-i} \mid k \ge 0\}$ is x.

Answers to True/False question **1.3:0.** (a) F. (b) F. (c) T. (d) T.

In the remaining two parts, let m_0 be any integer, and $m_1, m_2, \ldots$ any nonnegative integers < 10.

(c) Show that $\{\Sigma_{i=0}^{k}\, m_i\, 10^{-i} \mid k \geq 0\}$ is bounded above. (Suggestion: Show $m_0 + 1$ is an upper bound.) Thus this set will have a least upper bound, which we may call x.

(d) Let x be as in part (c), and $n_0, n_1, n_2, \ldots$ be constructed from this x as in (a) and (b). Show that if there are infinitely many values of k such that $n_k \neq 9$, then $n_k = m_k$ for all k. (Why is the restriction on cases where $n_k = 9$ needed?)

The remaining three exercises in this section go beyond the subject of the text, and examine the relationship of R with other ordered fields which do or do not satisfy the archimedean property. As their difficulty-numbers show, these should probably not be assigned in a non-honors course, though students whose curiosity is piqued by these questions might find them interesting to think about.

1.4:3. *Uniqueness of the ordered field of real numbers.* (**d**:5)

On p.21, Rudin mentions, but does not prove, that any two ordered fields with the least-upper-bound property are isomorphic. This exercise will sketch how that fact can be proved. For the benefit of students who have not had a course in Abstract Algebra, I begin with some observations generally included in that course (next paragraph and part (a) below).

If F is any field, let us define an element $n_F \in F$ for each integer n as follows: Let 0_F and 1_F be the elements of F called ‘‘0’’ and ‘‘1’’ in conditions (A4) and (M4) of the definition of a field. (We add the subscript F to avoid confusion with elements $0, 1 \in Z$.) For $n \geq 1$, once n_F is defined we recursively define $(n+1)_F = n_F + 1_F$; in this way n_F is defined for all nonnegative integers. Finally, for negative integers n we define $n_F = -(-n)_F$. (Note that in that expression, the ‘‘inner’’ minus is applied in Z, the ‘‘outer’’ minus in F.)

(a) Show that under the above definitions, we have $(m+n)_F = m_F + n_F$ and $(mn)_F = m_F\, n_F$ for all $m, n \in Z$.

(b) Show that if F is an *ordered* field, then we also have $m_F < n_F \Leftrightarrow m < n$ for all $m, n \in Z$. Deduce that in this case, the map $n \mapsto n_F$ is one-to-one.

The results of (a) and the first sentence of (b) above are expressed by saying that the map $n \mapsto n_F$ ‘‘respects’’ the operations of addition and multiplication and the order relation ‘‘<’’.

(c) Show that if F is an ordered field, and if for every rational number $r = m/n$ $(m, n \in Z,\ n \neq 0)$ we define $r_F = m_F / n_F \in F$, then $r \mapsto r_F$ is a well-defined one-to-one map $Q \to F$, which continues to respect addition, multiplication, and ordering (i.e., satisfies $(r+s)_F = r_F + s_F$, $(rs)_F = r_F\, s_F$, and $r_F < s_F \Leftrightarrow r < s$ for all $r, s \in Q$). Thus, Q is *isomorphic* as an ordered field to a certain subfield of F.

(The statement that the above map is ‘‘well-defined’’ means that the definition is *consistent*, in the sense that if we write a rational number r in two different ways, $r = m/n = m'/n'$ $(m, m'\ n, n' \in Z)$ then the two candidate values for r_F, namely m_F / n_F and m'_F / n'_F, are the same. We have to prove such a ‘‘well-definedness’’ result whenever we give a definition that depends on a choice of how to write something.)

(Remark: We constructed the map $Z \to F$ of (a) without assuming F ordered. Could we have done the same with the above map $Q \to F$? No. The trouble is that for some choices of F, the map $Z \to F$ would not have been one-to-one, hence starting with a rational number $r = m/n$, we might have found that $n_F = 0_F$ even though $n \neq 0$, and then m_F / n_F would not be defined.)

We will call an ordered field F *archimedean* if for all $x, y \in F$ with $x > 0_F$, there exists a positive integer n such that $n_F\, x > y$. Note that by the proof of Theorem 1.20(a), every ordered field with the least-upper-bound property is archimedean. If F is an archimedean ordered field, then for every $x \in F$ let us define $C_x = \{r \in Q \mid r_F < x\}$. (This set describes how the element $x \in F$ ‘‘cuts’’ Q in two; thus it is called ‘‘the cut in Q induced by the element $x \in F$’’.)

(d) Let F be an archimedean ordered field, and K an ordered field with the least-upper-bound property

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Answers to True/False question **1.4:0.** (a) F. (b) F. (c) T. (d) F. (e) T. (f) F.

(hence also archimedean). Let us define a map $f\colon F \to K$ by setting $f(x) = \sup\{r_K \mid r \in C_x\}$ for each $x \in F$. Show that f is a well-defined one-to-one map which respects addition, multiplication, and ordering. (Note that the statements $f(r+s) = f(r) + f(s)$ etc. must now be proved for all r, s in F, not just in Q.) Show, moreover, that f is the *only* one-to-one map $F \to K$ respecting addition, multiplication, and ordering. In other words, F is isomorphic as an ordered field, by a *unique* isomorphism, to a subfield of K.

(e) Deduce that if two ordered fields F and K both have the least-upper-bound property, then they are isomorphic as an ordered fields.

(Hint: For such F and K, step (d) gives maps $f\colon F \to K$ and $k\colon K \to F$ which respect the field operations and the ordering. Hence the composite maps $fk\colon K \to K$ and $kf\colon F \to F$ also have these properties. Now the identity maps $\mathrm{id}_K\colon K \to K$ and $\mathrm{id}_F\colon F \to F$, defined by $\mathrm{id}_K(x) = x$ $(x \in K)$ and $\mathrm{id}_F(x) = x$ $(x \in F)$ also respect the field operations and the ordering. Apply the uniqueness statement of (d) to each of these cases, and deduce that f and k are inverse to one another.)

Further remarks: It is not hard to write down order-theoretic conditions that a subset C of Q must satisfy to arise as a cut C_x in the above situation. If we define a ''cut in Q'' abstractly as a subset $C \subseteq Q$ satisfying these conditions, then we can show that if F is any ordered field with the least-upper-bound property, the set of elements of F must be in one-to-one correspondence with the set of all cuts in Q. If we know that there exists such a field F, this gives a precise description of its elements. If we do not, it suggests that we could construct such a field by defining it to have one element x_C corresponding to each cut $C \subseteq Q$, and defining addition, subtraction, and an ordering on the resulting set, $\{x_C \mid C \text{ is a cut in } Q\}$. After doing so, we might note that the symbol ''x'' is a superfluous place-holder, so the operations and ordering could just as well be defined on $\{C \mid C \text{ is a cut in } Q\}$. This is precisely what Rudin will do, though without the above motivation, in the Appendix to Chapter 1.

1.4:4. *Properties of ordered fields properly containing* R. (**d**:4)

Suppose F is an ordered field *properly* containing R.

(a) Show that for every element $\alpha \in F$ which does not belong to R, either (i) α is greater than all elements of R, (ii) α is less than all elements of R, (iii) there is a greatest element $a \in R$ that is $< \alpha$, but no least element of R that is $> \alpha$, or (iv) there is a least element that is $> \alpha$, but no greatest element of R that is $< \alpha$.

(b) Show that there will in fact exist infinitely many elements α of F satisfying (i), infinitely many satisfying (ii), infinitely many as in (iii) for *each* $a \in R$, and infinitely many as in (iv) for each $a \in R$. (Hint to get you started: There must be at least one element satisfying one of these conditions. Think about how the operation of multiplicative inverse will behave on an element satisfying (i), respectively on an element satisfying (iii) with $a = 0$.)

In particular, from the existence of elements satisfying (i), we see that F will be non-archimedean.

(c) Show that for every $\alpha \in F$ not lying in R there exists $\beta > \alpha$ such that no element $x \in R$ satisfies $\alpha < x < \beta$. (This shows that R is *not* dense in F.)

1.4:5. *Constructing a non-archimedean ordered field*. (**d**:4)

We will indicate here how to construct a non-archimedean ordered field F containing the field R of real numbers.

The elements of F will be the *rational functions* in a variable x, that is, expressions $p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials with coefficients in R, and q is not the zero polynomial. Unfortunately, though expressions $p(x)/q(x)$ are called rational ''functions'', they are not in general functions on the whole real line, since they are undefined at points x where the denominator is zero. We may consider each such expression as a function on the subset of the real line where its denominator is nonzero (consisting of all but finitely many real numbers); but we then encounter another problem: We want to consider rational functions such as $(x^2 - 1)/(x - 1)$ and $(x + 1)/1$ as the same; but they are not, strictly, since they have different domains.

There are technical ways of handling this, based on defining a rational function to be an ''equivalence class'' of such partial functions under the relation of agreeing on the intersections of their domains, or as an equivalence classes of pairs $(p(x), q(x))$ under an appropriate equivalence relation. Since the subject of equivalence classes is not part of the material in Rudin, I will not go into the technicalities here, but will simply say that we will consider two rational functions to ''be the same'' if they can be obtained from one another by multiplying and dividing numerator and denominator by equal factors, equivalently, if they agree wherever they are both defined; and will take for granted that the set of these elements form a field (denoted $R(x)$ by algebraists). We can now begin.

(a) Show that if q is a polynomial, then either q is ''eventually positive'' in the sense that

$$(\exists\, B\in R)\,(\forall\, r\in R)\ (r > B \Rightarrow q(r) > 0)$$

or q is ''eventually negative'', i.e.,

$$(\exists\, B\in R)\,(\forall\, r\in R)\ (r > B \Rightarrow q(r) < 0).$$

or $q = 0$. (Hint: look at the sign of the coefficient of the highest power of x in $q(x)$.)

(b) Deduce that if f is a rational function, then likewise either

$$(\exists\, B\in R)\,(\forall\, r\in R)\ (r > B \Rightarrow f(r) > 0),$$

or

$$(\exists\, B\in R)\,(\forall\, r\in R)\ (r > B \Rightarrow f(r) < 0).$$

or $r = 0$. Again, let us say in the first two cases that f is ''eventually positive'', respectively ''eventually negative''.

Given rational functions f and f', let us write $f < f'$ if $f' - f$ is eventually positive.

(c) Show that the above relation ''<'' makes the field of rational functions an ordered field F.

We shall regard the real numbers as forming a subfield of F, consisting of the *constant* rational functions $r/1$ $(r\in R)$. In particular, the sets of integers and of rational numbers also become subsets of F.

(d) Show that in F, the polynomial x (i.e., the rational function $x/1$, which is the function f given by $f(r) = r$) is $> n$ for all integers n. Thus, F is not archimedean.

We note a consequence:

(e) Deduce that the element $1/x\in F$ is positive, but is less than all positive rational numbers, hence less than all positive real numbers. (Thus, ''from the point of view of F'', the field of real numbers has a ''gap'' between 0 and the positive real numbers. It similarly has ''gaps'' between every real number and all the numbers above or below it.)

1.4:6. *A smoother approach to the archimedean property.* (**d**: 2)

Let F be an ordered field.

(a) Suppose A is a subset of F which has a least upper bound, $\alpha\in F$, and x is an element of F. Show that $\{a+x \mid a\in A\}$ has a least upper bound, namely $\alpha+x$.

(b) Suppose x is an element of F, and we let $A = \{nx \mid n\in Z\}$. Show that $\{a+x \mid a\in A\} = A$.

(c) Combining the results of (a) and (b), deduce that if x is a nonzero element of F, then the set $\{nx \mid n\in Z\}$ cannot have a least upper bound in F.

(d) Now suppose x is a positive element, and that F has the least upper bound property. Deduce from (c) that $\{nx \mid n\in Z\}$ is not bounded above; deduce from this that $\{nx \mid n\in J\}$ is not bounded above, where J denotes the set of positive integers, and deduce from this the statement of Theorem 1.20(a).

1.4:7. *An induction-like principle for the real numbers.* (**d**: 1, 2, 2)

Parts (a) and (b) below will show that some first attempts at formulating analogs of the principle of mathematical induction with real numbers in place of integers do not work. Part (c) gives a form of the principle that is valid.

Let symbols x, y, etc. denote nonnegative real numbers. Suppose that for each x, a statement $P(x)$

about that number is given, and consider the following four conditions:

(i) $P(0)$ is true.

(ii) For every x, if $P(y)$ is true for all $y < x$, then $P(x)$ is true.

(iii) For every x such that $P(x)$ is true, there exists $y > x$ such that for all z with $x \leq z \leq y$, $P(z)$ is true.

(iv) For all x, $P(x)$ is true.

(a) Show that (i) and (ii) do not in general imply (iv). (I.e., that there exist statements P for which (i) and (ii) hold but (iv) fails. Suggestion: Try the statement ‘‘$x \leq 1$’’.)

(b) Show, likewise, that (i) and (iii) (without (ii)) do not in general imply (iv).

(c) Show that (i), (ii) and (iii) together do imply (iv).

(Suggestion: If the set of nonnegative real numbers x for which $P(x)$ is false is nonempty, look at the greatest lower bound of that set.)

(Remark: Condition (i) is really a special case of (ii), and so could be dropped from (a) and (c), since with our variables restricted to nonnegative real numbers, ‘‘$P(y)$ holds for all $y < 0$’’ is vacuously true. But I include it to avoid requiring you to reason about the vacuous case.)

1.5. THE EXTENDED REAL NUMBER SYSTEM. (pp.11-12)

Relevant exercises in Rudin: None

Exercises not in Rudin:

1.5:0. *Say whether each of the following statements is true or false.*

(a) In the extended real numbers, $(+\infty)\cdot 0 = 1$.

(b) In the extended real numbers, $(-\frac{1}{2})\cdot(-\infty) = +\infty$.

The next exercise is in the same series as the last two exercises in the preceding section, and like them is tangential to the material in Rudin.

1.5:1. *Mapping a non-archimedean ordered field to the extended reals.* (**d**: 2. >**1.4:4**)

Suppose that F is an ordered field that properly contains R.

(a) Assuming the result of **1.4:4**, show that F can be mapped onto the extended real number system by a map f that carries each $\alpha \in R$ to itself, and which ‘‘respects’’ addition and multiplication, in the sense that it satisfies $f(\alpha+\beta) = f(\alpha) + f(\beta)$ and $f(\alpha\beta) = f(\alpha)\,f(\beta)$ except in the cases where these operations are not defined for the extended real numbers $f(\alpha)$ and $f(\beta)$. Briefly discuss the behavior of f in the latter cases.

(b) How does the f you have constructed behave with respect to the order-relation $<$?

1.6. THE COMPLEX FIELD. (pp.12-16)

Relevant exercises in Rudin:

1:R8. *C cannot be made an ordered field.* (**d**: 1)

1:R9. *C can be made an ordered set.* (**d**: 1,2)

Note that your answer to the final question, about the least-upper-bound property, requires either a proof that the property holds, or an argument showing why some set that is bounded above does not have a least upper bound.

1:R10. *Square roots in C.* (**d**: 2)

1:R11. *C = (positive reals)$\cdot$(unit circle).* (**d**: 1)

1:R12. *The n-term triangle inequality.* (**d**: 1)

1:R13. *An inequality on absolute values.* (**d**: 2)

1:R14. *An identity on the unit circle.* (**d**: 2)

1:R15. *When does equality hold in the Schwarz inequality?* (**d**: 3)

Exercise not in Rudin:

1.6:0. *Say whether each of the following statements is true or false.*

(a) C (the set of complex numbers, under the usual operations) is a field.

(b) For every complex number z, $\mathrm{Im}(\bar{z}) = \mathrm{Im}(-z)$.

(c) For all complex numbers w and z, $\mathrm{Re}(wz) = \mathrm{Re}(w)\,\mathrm{Re}(z)$.

1.7. EUCLIDEAN SPACES. (pp.16-17)

Relevant exercises in Rudin:

1:R16. *Solutions to* $|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$. (**d**: 2)

1:R17. *An identity concerning parallelograms.* (**d**: 2)

1:R18. *Vectors satisfying* $\mathbf{x} \cdot \mathbf{y} = 0$. (**d**: 1)

1:R19. *Solutions to* $|\mathbf{x} - \mathbf{a}| = 2\,|\mathbf{x} - \mathbf{b}|$. (**d**: 2)

The middle two lines of the above exercise should be understood to mean ‘‘$\{\mathbf{x} \mid |\mathbf{x}-\mathbf{a}| = 2\,|\mathbf{x}-\mathbf{b}|\} = \{\mathbf{x} \mid |\mathbf{x}-\mathbf{c}| = r\}$’’. Rudin actually gives the ‘‘solution’’ to this problem, but you have to prove that his solution has the asserted property.

Exercises not in Rudin:

1.7:0. *Say whether the following statement is true or false.*

(a) For all $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| \cdot |\mathbf{y}|$.

1.7:1. *Relations between* $|\mathbf{x}|$, $|\mathbf{y}|$ *and* $|\mathbf{x}+\mathbf{y}|$. (**d**: 3)

(a) Show that for any points $\mathbf{x}$ and $\mathbf{y}$ of R^k one has $|\mathbf{x}+\mathbf{y}| \geq |\mathbf{x}| - |\mathbf{y}|$ and $|\mathbf{x}+\mathbf{y}| \geq |\mathbf{y}| - |\mathbf{x}|$.

(b) Combining the above inequalities with that of Theorem 1.37(e), what is the possible set of values for $|\mathbf{x}+\mathbf{y}|$ if $|\mathbf{x}| = 3$ and $|\mathbf{y}| = 1$? If $|\mathbf{x}| = 1$ and $|\mathbf{y}| = 4$? Do there exist pairs of points $\mathbf{x}, \mathbf{y} \in R^2$ with $|\mathbf{x}| = 3$, $|\mathbf{y}| = 1$, and $|\mathbf{x}+\mathbf{y}|$ taking on each value allowed by your answer to the former question?

1.7:2. *A nicer proof of the Schwarz inequality for real vectors.* (**d**: 3)

Rudin’s proof of the Schwarz inequality is short, but messy. This exercise will indicate what I hope is a more attractive proof in the case of real numbers, and the next exercise will show how, with a bit of additional work, it can be extended to complex numbers. In the exercise after that, we indicate briefly still another version of these proofs which can be used if we consider the quadratic formula as acceptable background material

Although Rudin proves the Schwarz inequality on the page before he introduces Euclidean space R^k, we will here assume the reverse order, so that we can write our relations in terms of dot products of vectors.

Suppose $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are real numbers, and let us write $\mathbf{a} = (a_1, \ldots, a_k) \in R^k$, $\mathbf{b} = (b_1, \ldots, b_k) \in R^k$. Then $\mathbf{a}\,|\mathbf{b}| - |\mathbf{a}|\,\mathbf{b}$ is also a member of R^k. Its dot product with itself, being a sum of squares, is nonnegative. Expand the inequality stating this nonnegativity, using the distributive law for the dot product, and translate occurrences of $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{b} \cdot \mathbf{b}$ to $|\mathbf{a}|^2$ and $|\mathbf{b}|^2$. Assuming $\mathbf{a}$ and $\mathbf{b}$ both nonzero, you can now cancel a factor of $2\,|\mathbf{a}|\,|\mathbf{b}|$ from the whole formula and obtain an inequality close to the Schwarz inequality, but missing an absolute-value symbol. Putting $-\mathbf{a}$ in place of $\mathbf{a}$ in this inequality, you get a similar inequality, but with a sign reversed. Verify that these two inequalities are together equivalent to the Schwarz inequality.

The above derivation excluded the cases $\mathbf{a} = 0$ and $\mathbf{b} = 0$. Show, finally, that the Schwarz inequality

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Answers to True/False question **1.5:0.** (a) F. (b) T.

holds for trivial reasons in those cases.

1.7:3. *Extending the above result to complex vectors.* (**d**:3)

One can regard k-tuples of complex numbers as forming a complex vector space C^k; but it is not very useful to define the dot product $\mathbf{a}\cdot\mathbf{b}$ of vectors in this space as $\Sigma\, a_i\, b_i$, because this would not satisfy the important condition that $\mathbf{a}\cdot\mathbf{a} \neq 0$ for nonzero $\mathbf{a}$. So one instead defines $\mathbf{a}\cdot\mathbf{b} = \Sigma\, a_i\, \overline{b_i}$, and notes that for a nonzero vector $\mathbf{a}$, $\mathbf{a}\cdot\mathbf{a}$ is a positive real number by Theorem 1.31(e) (p.14). Thus, one can again define $|\mathbf{a}| = |\mathbf{a}\cdot\mathbf{a}|^{1/2}$. The above dot product satisfies most of the laws holding in the real case, but note two changes: First, though as before we have $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b})$, we now have $\mathbf{a}\cdot(c\mathbf{b}) = \bar{c}(\mathbf{a}\cdot\mathbf{b})$. Secondly, the dot product is no longer commutative. Rather, $\mathbf{b}\cdot\mathbf{a} = \overline{\mathbf{a}\cdot\mathbf{b}}$.

With these facts in mind, repeat the calculation of the preceding exercise for vectors $\mathbf{a}$ and $\mathbf{b}$ of complex numbers. You will get an inequality close to the one you first got in that exercise, except that in place of $\mathbf{a}\cdot\mathbf{b}$, you will have the expression $\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{b}+\overline{\mathbf{a}\cdot\mathbf{b}}) = \mathrm{Re}(\mathbf{a}\cdot\mathbf{b})$. To get around this problem, verify that for every complex number z there exists a complex number γ with $|\gamma| = 1$ such that $\gamma z = |z|$. Choosing such a γ for $z = \mathbf{a}\cdot\mathbf{b}$, verify that on putting $\gamma\,\mathbf{a}$ in place of $\mathbf{a}$ in your inequality, you get the Schwarz inequality. Again, give a separate quick argument for the case where $\mathbf{a}$ or $\mathbf{b}$ is zero.

1.7:4. *The Schwarz inequality via the quadratic formula.* (**d**:3)

The method of proving the real Schwarz inequality given in **1.7:2** requires us to remember one trick, ‘‘Take the dot product of $\mathbf{a}\,|\mathbf{b}| - |\mathbf{a}|\,\mathbf{b}$ with itself, and use its nonnegativity.’’ One can also prove the result with a slightly different trick: ‘‘Let t be a real variable, regard the dot product of $\mathbf{a} + t\mathbf{b}$ with itself as a real quadratic function of the real variable t, and use its nonnegativity.’’

Namely, expand that dot product in the form $a\,t^2 + b\,t + c$, note why the coefficients a, b and c are real, and recall that a function of that form has a change of sign if the discriminant $b^2 - 4\,a\,c$ is positive. Conclude that the discriminant must here be ≤ 0. Verify that that conclusion yields the real case of the Schwarz inequality (this time without special treatment of the situation where $\mathbf{a}$ or $\mathbf{b}$ is zero). Again, you can get the complex Schwarz inequality by applying the ideas of **1.7:3** to this argument.

The only difficulty is that since we are developing the properties of R from scratch, we should not assume without proof the above property of the discriminant! So for completeness, you should first prove that property. This is not too difficult. Namely, assuming the discriminant is positive, check by computation that if $a \neq 0$, the quadratic formula you learned in High School leads to a factorization of $a\,t^2 + b\,t + c$, and that this results in a change in sign. The case $a = 0$ can be dealt with by hand.

1.8. APPENDIX to Chapter 1. (Constructing *R* by Dedekind cuts.) (pp.17-21)

Relevant exercise in Rudin:

1:R20. *What happens if we weaken the definition of cut?* (**d**:3)

Exercises not in Rudin:

1.8:0. *Say whether each of the following statements is true or false.*

Here α, β denote cuts in Q, and r, s elements of Q.

(a) $\alpha+\beta = \{r+s : r\in\alpha,\ s\in\beta\}$.

(b) $\alpha\beta = \{rs : r\in\alpha,\ s\in\beta\}$.

(c) $-\alpha = \{-r : r\in\alpha\}$.

(d) $\alpha < \beta \Leftrightarrow \alpha$ is a proper subset of β.

(e) $r^* = \{s : s \leq r\}$.

1.8:1. *Some details of the proof of the distributive law for real numbers.* (**d**:3)

Verify the assertion in Step 7 of the proof of Theorem 1.19 (p.20) that multiplication of cuts satisfies

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Answers to True/False question **1.6:0.** (a) T. (b) T. (c) F. **Answer** to True/False question **1.7:0.** (a) T.

the distributive law for the following list of typical cases

(i) $\alpha < 0^*,\ \beta > 0^*,\ \gamma > 0^*,$ (iii) $\alpha > 0^*,\ \beta > 0^*,\ \beta+\gamma = 0^*,$

(ii) $\alpha < 0^*,\ \beta > 0^*,\ \beta+\gamma < 0^*,$ (iv) $\alpha = 0^*;$

or, for variety, the cases

(v) $\alpha > 0^*,\ \beta < 0^*,\ \gamma < 0^*,$ (vii) $\alpha < 0^*,\ \beta > 0^*,\ \beta+\gamma = 0^*,$

(vi) $\alpha > 0^*,\ \beta < 0^*,\ \beta+\gamma > 0^*,$ (vii) $\beta = 0^*.$

1.8:2. *The distributive law for the real numbers: another approach.* (**d**:4)

This exercise shows an alternative to the separate verification of the 27 cases of the distributive law in Step 7 of the proof of Theorem 1.19 (p.20). We begin with two preparatory steps:

(a) Suppose F is a set with operations of addition and multiplication satisfying axioms (A) and (M) on p.5 of Rudin. Show that F satisfies axiom (D) on p.6, i.e.,

(D) $$(\forall\, x,y,z\in F)\ \ x(y+z) = xy + xz$$

if and only if it satisfies

(D′) $$(\forall\, x,y,z,w\in F)\ \ (y+z+w = 0)\ \Rightarrow\ (xy+xz+xw = 0).$$

(Suggestion: First show that each of (D) and (D′) implies that $x(-y) = -(xy)$, and then prove with the help of this identity that each of (D) and (D′) implies the other.)

(b) Rudin noted in Step 6 that (D) (which in the case of real numbers we will here write $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$) held for positive real numbers. With multiplication extended to all real numbers as in Step 7, verify that (D) still holds when $\alpha = 0^*$, then that it holds when $\beta = 0^*$, then note that the case $\gamma = 0^*$ follows from the case $\beta = 0^*$ using commutativity of addition. Thus, we now know that it holds whenever $\alpha,\ \beta,\ \gamma \geq 0^*$. Verify also that the definition of multiplication of not-necessarily positive real numbers in Rudin's Step 7 implies the properties $\alpha(-\beta) = -(\alpha\beta) = (-\alpha)\beta$.

By part (a), in order to show that the multiplication we have defined is distributive for all real numbers, it will suffice to show that these satisfy condition (D′), i.e.,

$$(\forall\, \alpha,\beta,\gamma,\delta\in R)\ \ (\beta+\gamma+\delta = 0^*)\ \Rightarrow\ (\alpha\beta+\alpha\gamma+\alpha\delta = 0^*).$$

We will do this in two parts:

(c) Prove that if the above formula is true for all $\alpha \geq 0^*$, then it is true for all $\alpha\in R$.

(d) To prove the above formula in the case $\alpha \geq 0^*$, note that if $\beta,\gamma,\delta\in R$ satisfy $\beta+\gamma+\delta = 0^*$, then either two of $\beta,\ \gamma,\ \delta$ are $\geq 0^*$ and one is $\leq 0^*$, or two are $\leq 0^*$ and one is $\geq 0^*$. Since we know from Rudin's Step 4 that the addition of R is commutative, we can in each of these cases rename and rearrange terms so that the ‘‘two’’ referred to are α and β and the ‘‘one’’ is γ. Show that the result needed in the first case follows quickly from Rudin's Step 6, while the second follows easily from the first, using the identity $\alpha(-\beta) = -(\alpha\beta)$.

1.8:3. *A second round of cuts doesn't change R.* (**d**:3)

The constructions of this Appendix can be carried out starting with any ordered field F in place of Q; the result will be a set F' with an ordering, with two operations of addition and multiplication, and with a map $r \mapsto r^*$ of F into F'.

Show that if we start with $F = R$, the ordered field of real numbers, then F' will be isomorphic to R; more precisely, that the map $r \mapsto r^*$ will be an isomorphism (a one-to-one and onto map respecting the operations and the ordering). This shows that in a sense, the field of real numbers has ‘‘no gaps left to fill’’.

In doing this exercise, you may take for granted that the assertions Rudin makes in Step 8 of his construction remain valid in this context of a general ordered field F. (This in fact leaves just one nontrivial statement for you to prove. Make clear what it is before setting out to prove it.)

Answers to True/False question **1.8:0.** (a) T. (b) F. (c) F. (d) T. (e) F.

1.8:4. *Cuts don't work well on a nonarchimedean ordered field.* (**d**: 4)

Rudin notes in the middle of p.19 that the archimedean property of Q is used in proving that R satisfies axiom (A5) (additive inverses). Suppose F is an ordered field which does *not* have the archimedean property, and that (as in the preceding exercise) we carry out the construction of this Appendix, getting an ordered set F' with operations of addition and multiplication. Show that F' fails to satisfy axiom (A5).

1.8:5. *Proving* $r^*s^* = (rs)^*$. (**d**: 3)

In parts (a)-(c) below, let r and s be *positive* rational numbers. In those parts you will prove that these r and s satisfy $r^*s^* = (rs)^*$ i.e., assertion (b) of Step 8 in the construction of the real numbers (p.20). In part (d) you will look at the case where the factors are not both positive.

(a) Verify that r^*s^* and $(rs)^*$ contain the same *nonpositive* elements, and use the definitions given in Rudin to describe the *positive* elements each of them contains in terms of operations and inequalities in Q. Thus, it remains to prove that the sets of positive elements you have described are equal.

(b) Verify the inclusion $r^*s^* \subseteq (rs)^*$.

(c) To obtain the inclusion $(rs)^* \subseteq r^*s^*$, suppose p is a positive element of $(rs)^*$. From the fact that $p < rs$, deduce that there are rational numbers t_1 and t_2, both > 1, such that $rs/p = t_1 t_2$. (Suggestion: Show there is a t_1 such that $1 < t_1 < rs/p$, and then choose t_2 in terms of t_1.) As the analog of the first display on p.21 of Rudin, write $r' = r/t_1$, $s' = s/t_2$, and complete the proof as in Rudin, using multiplication instead of addition.

This completes the proof that $r^*s^* = (rs)^*$ for r and s positive. As with the axioms for multiplication in R, the remaining cases are deduced from this one. I will just ask you to do one of these:

(d) Deduce from the case proved above that Rudin's assertion (b) also holds when $r < 0$ and $s > 0$.

Chapter 2. Basic Topology.

2.1. FINITE, COUNTABLE, AND UNCOUNTABLE SETS. (pp.24-30)

Relevant exercises in Rudin:

2:R1. *The empty set is everywhere.* (**d**: 1)

2:R2. *The set of algebraic numbers is countable.* (**d**: 3)

For this exercise, the student should take as given the result (generally proved in a course in Abstract Algebra) that a polynomial of degree n over a field has at most n roots in that field.

2:R3. *Not all real numbers are algebraic.* (**d**: 1. >**2:R2**)

2:R4. *How many irrationals are there?* (**d**: 2)

Exercises not in Rudin:

2.1:0. *Say whether each of the following statements is true or false.*

(a) If $f: X \to Y$ is a mapping, and E is a subset of Y containing exactly one element, then the subset $f^{-1}(E)$ of X also contains exactly one element.

(b) If Y is a set and $\{G_\alpha \mid \alpha \in A\}$ is a family of subsets of Y, then $\bigcup_{\alpha \in A} G_\alpha$ is a subset of Y.

(c) Every proper subset of J (the set of positive integers) is finite.

(d) The range of any sequence is at most countable.

(e) The set of rational numbers r satisfying $r^2 < 2$ is countable.

(f) Every infinite subset of an uncountable set is uncountable.

(g) Every countable set is a subset of the integers.

(h) Every subset of the integers is at most countable.

(i) Every finite set is countable.

(j) The union of any collection of countable sets is countable.

(k) If A and B are countable sets, then $A \cup B$ is countable.

(l) If A is countable and B is any set, then $A \cap B$ is countable.

(m) If A is countable and B is any set, then $A \cap B$ is at most countable.

2.1:1. *An infinite image of a countable set is countable.* (**d**: 2)

Suppose E is a countable set, and f is a function whose domain is E and whose image $f(E)$ is infinite. Show that $f(E)$ is countable. (Hint: The proof will be like that of Theorem 2.8, but this time, take $n_1 = 1$, and for each $k > 1$, assuming $n_1, \ldots, n_{k-1}$ have been chosen, let n_k be the least integer such that $x_{n_k} \notin \{x_{n_1}, \ldots, x_{n_{k-1}}\}$. To do this you must note why there is at least one such n_k.)

2.1:2. *Functions and cardinalities.* (**d**: 2, 1, 1, 1)

Suppose A and B are sets, and $f: A \to B$ a function.

(a) Assume A and B infinite. We can divide this situation into four cases, according to whether A is countable or uncountable and whether B is countable or uncountable. Show that if f is one-to-one, then three of these four cases can occur, but one cannot. To do this, you must give examples of three cases, and a proof that the fourth cannot occur. (Hint: Some or all of your examples can be of trivial sorts; e.g., using functions that don't move anything, but satisfy $f(x) = x$ for all x.) Express your nonexistence result as an implication saying that if $f: A \to B$ is one-to-one, and a certain one of A or B has a certain property, then the other has a certain property.

(b) If we don't assume A and B infinite, then each of these sets can be finite, countable, or uncountable, giving $3 \times 3 = 9$ rather than 4 combinations. Again, for f one-to-one, certain of these 9 cases are possible and certain impossible. I won't ask your to prove your assertions (since your understanding of the consequences of finiteness is largely intuitive, and a course in set theory, not Math 104, is where you will learn the theory that will make it precise); but make a 3×3 chart showing which cases can occur, and which cannot. (Label the rows with the properties of A, in the order ''finite; countable; uncountable'', the columns with the properties of B in the same order, and use $\surd$ and $\times$ for ''possible'' and ''impossible''. If you don't know the answer in some case, use ''?'')

(c) As in (a), assume A and B infinite, so that we have four cases depending on whether each is countable or uncountable; but now suppose f is onto, rather than one-to-one. Again, give examples showing that three cases can occur, but show that the fourth cannot. (Hint: Use **2.1:1**.) Again, express your nonexistence result as an implication.

(d) Analogously to part (b), modify part (c) by not assuming A and B infinite, and make a 3×3 chart showing which cases can occur, and which cannot.

2.1:3. *Cardinalities of sets of functions.* (**d**: 3)

Suppose A is a countable set, and B is a set (finite, countable, or infinite), which contains at least two elements. Show that there are uncountably many functions $A \to B$.

Suggestion: Consider first the case where $A = J$ and $B = \{0,1\}$, and convince yourself that this case is equivalent to a result Rudin has proved for you.

This would probably not be a good problem to assign, because students who had seen some set theory might have a big advantage over those who hadn't. But it is a good one for students to think about if they haven't seen these ideas.

2.2. METRIC SPACES. (pp.30-36)

Relevant exercises in Rudin:

2:R5. *Find a bounded set with 3 limit points.* (**d**: 2)

2:R6. *Properties of* $\{$*limit points of* $E\}$. (**d**: 2)

2:R7. *Closures of unions versus unions of closures.* (**d**: 2)

In the last sentence of this exercise, ''this inclusion can be proper'' means that there are some choices of metric space and subsets A_i such that the union of closures shown at the end of (b) is a *proper* subset of $\overline{B}$. (If you're unsure what ''proper subset'' means, use the index!)

2:R8. *Limits points of closed and open sets.* (**d**: 2)

2:R9. *Basic properties of the interior of a set.* (**d**: 2)

2:R10. *The metric with* $d(p,q) = 1$ *for all* $p \neq q$. (**d**: 2)

(But the last sentence of this exercise refers to the concept of compactness, and so requires §2.3.)

2:R11. *''Which of these five functions are metrics?''* (**d**: 2)

In this exercise you must, for each case, either prove that the properties of a metric are satisfied, or give an example showing that one of these properties fails.

Exercises not in Rudin:

2.2:0. *Say whether each of the following statements is true or false.*

(a) Every unbounded subset of R is infinite.

(b) Every infinite subset of R is unbounded.

(c) If E is a subset of a metric space X, then every interior point of E is a member of E.

(d) If E is a bounded subset of a metric space X, then every subset of E is also bounded.

(e) Q is a dense subset of R.

(f) R is a dense subset of C.

(g) If E is a subset of a set X, then $(E^c)^c$ (the complement of the complement of E in X) is E.

(h) If E is an open subset of a metric space X, then every subset of E is also an open subset of X.

(i) If E is an open subset of a metric space X, then E^c is a closed subset of X.

(j) If E is a subset of a metric space X, and E is not open, then it is closed.

(k) If Y is a subset of a metric space X, and $\{G_\alpha\}$ is a family of subsets of Y that are open relative to Y, then $\bigcup G_\alpha$ is also open relative to Y.

(l) If E is a subset of a metric space X and p is a limit point of E, then there exists $q \in E$ such that $q \neq p$ and such that q belongs to every neighborhood of p in X.

(m) If E is a subset of a metric space X and p is a limit point of E, then for every neighborhood N of p in X, there exists $q \in E \cap N - \{p\}$.

(n) The union of any two convex subsets of R^k is convex.

(o) The intersection of any family of convex subsets of R^k is convex.

2.2:1. *Possible distances among 3 points.* (**d**: 1. >**1.7:1**)

(a) Show that for any points p, q and r of a metric space, one has $d(p,r) \geq d(p,q) - d(q,r)$ and $d(p,r) \geq d(q,r) - d(p,q)$.

(b) Combining the above inequalities with that of Definition 2.15(c), what is the possible set of values for $d(p,r)$ if $d(p,q) = 3$ and $d(q,r) = 1$? If $d(p,q) = 1$ and $d(q,r) = 4$? For each value $d(p,r) = c$ allowed by your answer to the former question, do there in fact exist points p, q and r in some metric space X such that $d(p,q) = 3$, $d(q,r) = 1$, and $d(p,r) = c$? Hint: Use **1.7:1**.

2.2:2. *A characterization of open sets.* (**d**: 1)

Show that a subset E of a metric space X is *open* if and only if it is the union of a set of neighborhoods.

Answers to True/False question **2.1:0.** (a) F. (b) T. (c) F. (d) T. (e) T. (f) F. (g) F. (h) T. (i) F. (j) F. (k) T. (l) F. (m) T.

2.2:3. *A characterization of the closure of a set.* (**d**: 1)

Let E be a subset of a metric space X. Show that $\overline{E} = \{x \in X \mid (\forall \varepsilon > 0)(\exists y \in E)\ d(y,x) < \varepsilon\}$. (I prefer this to the definition of closure that Rudin gives, since the splitting of the points of $\overline{E}$ into two sorts, those in E and the limit points, seems to me unnatural.)

2.2:4. *A characterization of limit points.* (**d**: 1)

Let X be a metric space and E a subset. Show that a point $p \in X$ is a limit point of E if and only if every open subset $G \subseteq X$ which contains p contains some point of E other than p. (This differs from part (b) of Definition 2.18 only in the replacement of ''neighborhood of p'' by ''open subset of X containing p''.)

2.2:5. *Open and closed subsets of open and closed sets.* (**d**: 2)

(a) Suppose E is an *open* subset of a metric space X, and F is a subset of E. Show that F is open relative to E if and only if it is open as a subset of X.

(b) Suppose E is a *closed* subset of a metric space X, and F is a subset of E. Show that F is closed relative to E if and only if it is closed as a subset of X.

(c) Suppose E is a *open* subset of a metric space X, and F is a subset of E which is *closed* relative to E. Show by an example that F need not, in general, either be open or closed as a subset of X. (Give a specific metric space X and specific subsets E and F.)

(d) Give a similar example showing that for a closed subset E, a relatively open subset F of E need neither be open nor closed in X.

2.2:6. *The boundary of a subset of a metric space.* (**d**: 2)

Let X be a metric space, and E a subset of X. One defines the *boundary* of E to be the set ∂E of all points $x \in X$ such that every neighborhood of x contains at least one point of E and at least one point of E^c. (In saying ''at least one point'', we do not exclude the point x itself.)

(a) Show that $\overline{E} = E \cup \partial E$.

(b) Deduce that E is closed if and only if $\partial E \subseteq E$.

(c) Show that E is open if and only if $\partial E \subseteq E^c$.

(d) Deduce that E is both open and closed if and only if $\partial E = \varnothing$.

2.2:7. *Equivalent formulations of boundedness.* (**d**: 2)

Let E be a nonempty subset of a metric space X. Show that the following conditions are equivalent:

(a) E is bounded. (Definition 2.18(i).)

(b) For every point $q \in X$ there exists a real number M such that for all $p \in E$, $d(p,q) < M$.

(c) There exists a real number M and a point $q \in E$ such that for all $p \in E$, $d(p,q) < M$.

(d) There exists a real number M such that for all $p, q \in E$, $d(p,q) < M$.

(Warning: The ''M''s of statements (a)-(d) will not necessarily be the same.)

2.2:8. *Another description of the closure of a set.* (**d**: 1)

Let E be a subset of a metric space X. Show that $\overline{E}$ equals the intersection of all closed subsets of X containing E.

2.2:9. *The closure of a bounded set is bounded.* (**d**: 1)

Let E be a bounded subset of a metric space X. Show that $\overline{E}$ is also bounded.

2.2:10. *Finding bounded perfect subsets of perfect sets.* (**d**: 3. >**2.2:9**)

Let X be a metric space.

(a) Show that if E is a perfect subset of X and A is an open subset of X, then $\overline{E \cap A}$ is perfect.

(b) Deduce that if X has a nonempty perfect subset, then it has a bounded nonempty perfect subset.

Answers to True/False question **2.2:0.** (a) T. (b) F. (c) T. (d) T. (e) T. (f) F. (g) T. (h) F. (i) T. (j) F. (k) T. (l) F. (m) T. (n) F. (o) T.

2.2:11. *Modifying a metric to get another metric.* (**d**: 2,2,4, cf. **2:R11**)

(a) Suppose X is a metric space, with metric d. Show that the function d' given by $d'(x,y) = d(x,y)^{1/2}$ is also a metric on X, and that the same sets are open in X under the metric d' as under the metric d.

(b) Will the same be true of the function d'' given by $d''(x,y) = d(x,y)^2$?

(c) For what functions f from the nonnegative real numbers to the nonnegative real numbers is it true that for every metric d on a set X, the function d^f defined by $d^f(x,y) = f(d(x,y))$ is also a metric on X?

2.2:12. *A non-closed set has no largest closed subset.* (**d**: 2)

Let E be a subset of a metric space X. Theorem 2.27(c) shows that even if E is not closed, there is a smallest closed subset of X containing E; i.e., a closed subset which contains E and is contained in all closed subsets which contain E. However –

(a) Show that if E is not closed, then there does not exist a largest closed subset contained in E. (Hint: If F is any closed subset contained in E, show that by bringing in one more point one can get a larger closed subset, still contained in E.)

(b) Exercise **2:R9**(*c*) (p.43) shows that there is always a largest open subset of X contained in E. Will there in general exist a smallest open subset of X containing E?

2.2:13. *Reconciling Rudin's two uses of ''dense subset''.* (**d**: 2)

On p.9, in the sentence following Theorem 1.20, Rudin implicitly defines a subset $E \subseteq R$ to be ''dense'' if it has the property

(i) For all $x, y \in R$ with $x < y \in R$, there exists $p \in E$ such that $x < p < y$.

On the other hand, on p.32, Definition 2.18(j), a subset E of a general metric space X is defined to be ''dense'' if

(ii) Every point of X is a limit point of E or a point of E.

Prove that these uses of the word are consistent, by showing that a subset $E \subseteq R$ satisfies (i) if and only if it satisfies (ii).

2.2:14. *The n-adic metric on* Z. (**d**: 2)

Let $n > 1$ be a fixed integer.

For any nonzero integer s, let $e_n(s)$ be the largest integer a such that s is divisible by n^a. Now define a function d_n on pairs of integers by letting $d_n(s,t) = n^{-e_n(s-t)}$ if $s \neq t$, and letting it be 0 if $s = t$. Show that d_n is a metric on Z. In fact, in place of condition (c) of Definition 2.15, prove

(c') $$d_n(p,q) \leq \max(d_n(p,r),\, d_n(r,q)),$$

and then show that $(c') \Rightarrow (c)$.

(When n is a prime number p, the metric d_p is important in number theory, where it is called the ''p-adic metric'' on Z. Condition (c'), known as ''the ultrametric inequality'', is not satisfied by most metric spaces; in particular it does not hold in R.)

2.2:15. *Iterated limit sets.* (**d**: 4)

If E is a subset of a metric space X, let us (in this exercise) write $L(E)$ for the set of all limit points of E. We shall write $L^2(E)$ for $L(L(E))$, $L^3(E)$ for $L(L^2(E))$, etc.; $L^0(E)$ will denote E itself.

Show that for every positive integer n there exists a subset E of some metric space X such that $L^{n-1}(E) \neq \emptyset$, but $L^n(E) = \emptyset$. (Note: This can be done using $X = R$, but you may, if you prefer, give a construction in some other metric space.)

2.2:16. *Limit points described in terms of closures.* (**d**: 1)

Let X be a metric space, E a subset of X, and p a point of X. Show that p is a limit point of E if and only if $p \in \overline{E - \{p\}}$.

2.2:17. $\{q \in X : d(p,q) \leq r\}$. (**d**: 1,2)

Let X be a metric space. For every point p of X and positive real number r, let

$$V_r(p) \;=\; \{q \in X : d(p,q) \leq r\}.$$

(a) Show that $V_r(p)$ is closed, and that

$$N_r(p) \;\subseteq\; \overline{N_r(p)} \;\subseteq\; V_r(p).$$

(b) Give four examples of a metric space X, a point p, and a positive real number r, which together exhibit all four possible combinations of equality and inequality in the above displayed line; i.e., a case where all three sets are equal, a case where there is equality at the first ''$\subseteq$'' but not at the second; a case where there is equality at the second but not at the first, and a case where all three sets are distinct.

2.2:18. *Not every finite metric space embeds in an* R^k. (**d**: 1,3,4)

Let X be a 4-element set $\{w,x,y,z\}$, and let d be the metric on X under which the distance from w to each of the other points is 1, and the distance between any two of those points is 2.

(a) Verify that the above conditions do indeed determine a metric on X.

(b) Show that no function f of X into a space R^k is distance-preserving, i.e., satisfies $d(f(p),f(q)) = d(p,q)$ for all $p,q \in X$.

(c) The above example has the property that every 3-point subset of X can be embedded (mapped by a distance-preserving map) into a space R^k for some k, but the whole 4-point space cannot be so embedded for any k. Can you find a 5-point metric space, every 4-point subset of which can be so embedded but such that the whole 5-point space cannot?

2.2:19. *An infinite metric space has uncountably many open sets.* (**d**: 4,2)

Let X be an infinite metric space.

(a) Show that X has a countable family of pairwise disjoint neighborhoods; i.e., that there exist points p_i and positive real numbers r_i $(i \in J)$ such that whenever $i \neq j$, we have $N_{r_i}(p_i) \cap N_{r_j}(p_j) = \varnothing$.

(Remark: Looking at the case where X is the subset $\{1/n : n \in J\} \cup \{0\} \subseteq R$, you will find that if you take any of the p_i equal to 0, you can't complete the construction. Suggested step to get around this problem: Prove that given any infinite subset $E \subseteq X$ and any two distinct points x and y of X, at least one of x and y has a neighborhood that misses infinitely many points of E; and if you take a neighborhood of any smaller radius, there will be infinitely many points of E not in its closure.)

(b) Deduce that X has uncountably many open sets. (Hint: Associate to every sequence (s_i) of 0's and 1's the union of those $N_{r_i}(p_i)$ for which $s_i = 1$.)

2.2:20. *Iterated interior and closure operations.* (**d**: 1,3,2,1,4. >**2:R9**)

Let E be a subset of a metric space X. We shall examine here how many different sets we can get by successively applying the closure and interior operators to E.

(a) Show that $\overline{\overline{E}} = \overline{E}$, and that $(E^\circ)^\circ = E^\circ$. (Hint: For the case of closure, make use of Theorem 2.2.7. For the case of interior, you may assume the assertions of **2:R9**.)

It follows that if we start with a set and apply some sequence of closure and interior operators to it (e.g., take the closure of the interior of the interior of the closure of the closure of E), the application of one or the other of these operators more than once in succession gives nothing more than applying it once (e.g., the set just described is simply the closure of the interior of the closure of E); so anything we can get, we can get at least as simply by applying closure and interior alternately. This could still, in principle, lead to infinitely many different sets; but the next result limits further the distinct sets we can get.

(b) Show that $\overline{(\overline{E}^\circ)}{}^\circ = \overline{E}{}^\circ$.

(c) Deduce from (b) that $\overline{(\overline{E^\circ})^\circ} = \overline{E^\circ}$. Again you may assume the results of **2:R9**.

(d) Deduce from (b) and (c) that starting with E and applying closure and interior operators, one can get at most 7 distinct sets (counting E itself).

(e) Show by example that a certain subset $E \subseteq R$ does indeed yield 7 distinct sets under these operations.

(Some of you might find it easier to keep track of the order of operations if you write cl(E) in place of $\overline{E}$ and int(E) in place of $E°$. As examples showing both the advantages and disadvantages of this notation, the equations of (b) and (c) would become

$$\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(E)))) = \mathrm{int}(\mathrm{cl}(E)) \qquad \text{and} \qquad \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(E)))) = \mathrm{cl}(\mathrm{int}(E)).$$

If you decide to use this notation, do so consistently throughout this exercise.)

2.2:21. *Iterated closures and complements.* (**d**: 1,1,3,3. >**2.2:20**)

This exercise continues the ideas of **2.2:20**. As in that exercise, let E be a subset of a metric space X.

(a) Show that $E°$ can be obtained by applying to E some combination of the operations of closure and complement. (You may assume **2:R9**(d).)

(b) Deduce that every subset of X that can be obtained from E using a sequence of the operations closure and interior can also be obtained using a sequence of the operations closure and complement.

(c) Deduce from (b) and the results of **2.2:20** that starting with E and applying closure and complement operators, one can get at most 14 distinct sets (counting E itself).

(e) Show by example that a certain subset $E \subseteq R$ does indeed yield 14 distinct sets under these operations.

(f) Does there exist a nonempty metric space X and a subset E such that some set obtained from E using the operator of closure and an even number of applications of the complement operation is equal to a set obtained from E using the operator of closure and an odd number of applications of the complement operation?

2.2:22. *Some questions on relative closures and interiors.* (**d**: 2)

Suppose X is a metric space and $Y \subseteq X$ a subset. For any $E \subseteq Y$, let us write $\mathrm{cl}_X(E)$ for the closure of E in X, and $\mathrm{cl}_Y(E)$ for the closure of E relative to Y; and similarly, $\mathrm{int}_X(E)$ and $\mathrm{int}_Y(E)$ for the interior of E in X and relative to Y respectively.

Determine which of the following statements are true whenever X and Y are as above, and E and F are two subsets of Y:

(a) $\mathrm{cl}_Y(E) = \mathrm{cl}_Y(F) \Rightarrow \mathrm{cl}_X(E) = \mathrm{cl}_X(F)$.

(b) $\mathrm{cl}_X(E) = \mathrm{cl}_X(F) \Rightarrow \mathrm{cl}_Y(E) = \mathrm{cl}_Y(F)$.

(c) $\mathrm{int}_Y(E) = \mathrm{int}_Y(F) \Rightarrow \mathrm{int}_X(E) = \mathrm{int}_X(F)$.

(d) $\mathrm{int}_X(E) = \mathrm{int}_X(F) \Rightarrow \mathrm{int}_Y(E) = \mathrm{int}_Y(F)$.

In any case(s) where the assertion is true, you should give a proof; the easiest way is show that the sets on the right can be constructed from those on the left in a way that doesn't depend on E or F. In the case(s) where the assertion is false, you should give a counterexample.

2.3. COMPACT SETS. (pp.36-40)

Relevant exercises in Rudin:

2:R12. $\{1/n\} \cup \{0\}$ *is compact.* (**d**: 2)

You can do this exercise as soon as you have read the definition of compactness. (You do not have to have read the Heine-Borel Theorem, which it tells you *not* to use in the proof.)

2:R13. *A compact set whose limit-set is countable.* (**d**: 3)

Of course, you need to prove that the set you give is compact, and justify your assertion as to what are its limit points.

2:R14. *An open cover of* $(0,1)$ *having no finite subcover.* (**d**: 1)

I've rated this **d**: 1 because the example is simple; but it might be difficult for a student to find if no similar examples have been pointed out.

2:R15. *Theorem 2.36 needs both ''closed'' and ''bounded''.* (**d**: 2)

Note that this exercise asks for two examples: one for ''closed'' and one for ''bounded''.

2:R16. *A closed bounded subset of Q need not be compact.* (**d**: 1)

2:R17. *Properties of* $\{x \in [0,1] \mid$ *the decimal expansion of* x *has only* 4*'s and* 7*'s*$\}$. (**d**: 3)

Another important result on compact sets is **2:R26**. It is one of a group of exercises involving the concept of *separable metric space*, defined in **2:R22**. I have classified these as a ''section'', **2.6**, at the end of this chapter, but this exercise (and those it depends on, see notes on that section below) could be done at this point.

Exercises not in Rudin:

2.3:0. *Say whether each of the following statements is true or false.*

(a) If we regard the set of open segments $\{(x+1, x-1) \mid x \in [-10, 10]\}$ as an open covering of the subset $[-10, 10]$ of the metric space R, then the set of open segments $\{(x+½, x-½) \mid x \in [-10, 10]\}$ is a subcovering.

(b) If we regard the set of open segments $\{(x+1, x-1) \mid x \in [-10, 10]\}$ as an open covering of the subset $[-10, 10]$ of the metric space R, then the set of open segments $\{(x+1, x-1) \mid x = -10, -9, ..., 0, ..., 9, 10\}$ is a subcovering.

(c) If we regard the set of open segments $\{(x+1, x-1) \mid x \in [-10, 10]\}$ as an open covering of the subset $[-10, 10]$ of the metric space R, then the set of open segments $\{(x+1, x-1) \mid x = -10, -8, ..., 0, ..., 8, 10\}$ is a subcovering.

(d) If K is a subset of a metric space, and some open covering of K has a finite subcovering, then K is compact.

2.3:1. *A union of finitely many compact sets is compact.* (**d**: 2)

Show that if $E_1, \ldots, E_n$ are compact subsets of a metric space X, then their union $E_1 \cup \ldots \cup E_n$ is also compact.

2.3:2. *Covering a compact set by neighborhoods that don't overlap too much.* (**d**: 3)

Let X be a compact metric space, and ε positive real number. Show that there exists a subset $S \subseteq X$ such that the sets $N_\varepsilon(s)$ $(s \in S)$ form a cover of X, and such that the distance between any two points of S is $\geq \varepsilon/2$.

(Suggestion: First find a finite set T such that the sets $N_{\varepsilon/2}(s)$ $(s \in T)$ form a cover of X. Then get a subset $S \subseteq T$ such that the distance between any two points of S is $\geq \varepsilon/2$, but such that no larger subset of T has that property. Then show that the $N_\varepsilon(s)$ $(s \in S)$ must cover X.)

(Remark: With the help of the Axiom of Choice, which one learns about in a course in set theory, or the consequence thereof called Zorn's Lemma, one can prove the same result without the assumption that X is compact. When X is compact, the set S obtained as above will be finite; for noncompact X, this is not generally so. However, unless your instructor tells you the contrary, you should use only the tools assumed in Rudin, in which case a proof using the Axiom of Choice or Zorn's Lemma is not an option.)

2.3:3. *If you pack too many points into a compact set, they get crowded.* (**d**: 3)

Suppose K is a compact metric space and ε a positive real number. Show that there is a positive integer N such that every set of N points of K includes at least two points of distance $< \varepsilon$ apart.

(Hint: Take N to be greater than the number of sets in some covering of K by neighborhoods of radius $\varepsilon/2$.)

2.3:4. *Another finiteness property of coverings of compact sets.* (**d**: 4. >**2:R26**)

Let X be a metric space. If $\{G_\alpha\}_{\alpha \in I}$ is a covering of X and $\beta \in I$, let us say that β is an *essential* index for this covering if $\{G_\alpha\}_{\alpha \in I, \alpha \neq \beta}$ is *not* a covering of X. Show that X is compact if and only if for every covering $\{G_\alpha\}$ of X there are only finitely many essential indices.

(Suggestion for the hard direction: If X is not compact, use **2:R26** to get a countable set S without limit points, and verify that the subsets of X gotten by removing from X all but one point of S form an open covering. It would be interesting to find a proof that does not use **2:R26**.)

2.3:5. *Redundant coverings of compact sets*. (**d**: 2)

Let K be a compact subset of a metric space, and $\{G_\alpha\}_{\alpha\in I}$ a covering of K with the property that every $p\in K$ belongs to at least *two* sets in this covering. Show that $\{G_\alpha\}_{\alpha\in I}$ has a finite subcovering with the same property.

2.3:6. R *is closed in any metric space*. (**d**: 3)

Suppose X is a metric space having the real line R as a subspace; i.e., such that R is a subset of X, and the metric on R induced by that of X is the standard metric $d(r,s) = |r-s|$. Show that R is closed in X. (Hint: Points close to each other in R belong to a compact subset.)

2.4. PERFECT SETS. (pp.41-42)

Relevant exercises in Rudin:

2:R18. *Can a perfect set be quite irrational?* (**d**: 4)

This exercise logically requires only the definition of ''perfect set'' in **2.2** and the material of Chapter 1. However, it may help to think about **2:R17** first, for inspiration.

2:R30. *Baire's Theorem: a property of countable closed coverings of* R^k. (**d**: 4)

Exercises not in Rudin:

2.4:0. *Say whether each of the following statements is true or false*.

(a) If E is a perfect subset of a metric space X, and F is a closed subset of E, then F is perfect.

(b) The Cantor set is countable.

2.4:1. *A more explicit proof of Theorem 2.43*. (**d**: 3)

Let P be as in the hypothesis of Theorem 2.43. Give a proof of that theorem, as sketched below, which obtains explicitly an uncountable subset of P, rather than getting a contradiction from the assumption that P is countable:

As the ''0th'' step, you will choose any neighborhood $V\subseteq X$ having nonempty intersection with P. At the next step, find neighborhoods V_0 and V_1, each of which has nonempty intersection with P and has closure contained in V, and such that those closures $\overline{V_0}$ and $\overline{V_1}$ are disjoint. Then find neighborhoods V_{00} and V_{01} in V_0, and V_{10} and V_{11} in V_1, with similar properties, and so on. Now show that there are uncountably many chains $\overline{V} \supset \overline{V_{a_1}} \supset \overline{V_{a_1 a_2}} \supset \dots$, that the intersection of the sets comprising any such chain contains a point of P, and that the points so obtained are distinct.

(This shows not only that P is uncountable, but that it has at least the cardinality of the set of all sequences of 0's and 1's. When one knows some set theory one sees that this is a stronger statement. Incidentally, one can add to the construction the condition that each $\overline{V_{a_1 \dots a_n}}$ has radius $< 1/n$, and conclude that the intersections one gets are single points. The set of these points will be a perfect set contained in P whose points correspond in a natural way to the points of the Cantor set.)

2.2:10 (given under **2.2** above) might alternatively be assigned in this section.

2.4:2. *Generalizing the idea of Theorem 2.43*. (**d**: 3. >**2.2:10**)

Let X be a metric space and E a subset.

(a) Show if E is perfect, compact and nonempty, then E is uncountable. (Since X is not assumed to be R^k, you cannot get this from Theorem 2.43; but I suggest you try to adapt the proof of that theorem – either the proof in Rudin, or the variant proof indicated in **2.4:1** above.)

Unfortunately, part (a) above does not subsume Theorem 2.43, since it assumes X compact, which R^k is not. But one can deduce that theorem from it:

Answers to True/False question **2.3:0.** (a) F. (b) T. (c) F. (d) F.

(b) Deduce Theorem 2.43 from part (a) above with the help of Theorem 2.41 and **2.2:10**.

2.4:3. *Two metrics on the Cantor set.* (**d**: 2,4)

Recall that the Cantor set is constructed in 2.44 as the intersection of a chain of sets $E_1 \supset E_2 \supset E_3 \supset \ldots$, where each set E_n is a union of 2^n intervals in $[0,1]$. Let us say that two points p, q of the Cantor set are ‘‘together at the nth stage’’ if p and q belong to the same interval in E_n. Let us define $t(p,q)$ to be the greatest integer n such that p and q are together at the nth stage, or $+\infty$ if $p = q$.

(a) Show that the function $d(p,q) = 2^{-t(p,q)}$ (defined to be 0 if $p = q$) is a metric on the Cantor set, and satisfies the ultrametric inequality (mentioned earlier in **2.2:14**):

$$d(p,q) \leq \max(d(p,r),\, d(r,q)).$$

(b) Show that the same subsets of the Cantor set are open with respect to this metric d as with respect to the ordinary distance-function $|x-y|$ of R.

2.5. CONNECTED SETS. (pp.42-43)

Relevant exercises in Rudin:

2:R19. *A connected space of at least two points is uncountable.* (**d**: 3)

More details on Rudin’s hint for part (d): Take $p_0, p_1 \in X$ and use part (c) to show that every positive real number $< d(p_0,p_1)$ has the form $d(p_0,q)$. Deduce that there are uncountably many $q \in X$.

2:R20. *Closures and interiors of connected sets.* (**d**: 2)

2:R21. *Convex subsets of R^k are connected.* (**d**: 3)

Exercises not in Rudin:

2.5:1. *A characterization of connectedness.* (**d**: 2)

(a) Show that a subset E of a metric space X is connected if and only if the only subsets of E which are both open and closed relative to E are E and $\varnothing$.

(b) Deduce that a closed subset of a metric space is connected if and only if it cannot be written as the union of two disjoint closed subsets, and that an open subset of a metric space is connected if and only if it cannot be written as the union of two disjoint open subsets.

2.5:2. *A set with enough connected subsets is connected.* (**d**: 2)

Let E be a subset of a metric space X. Show that E is connected if and only if for every two points $p, q \in E$, there is a connected subset $A \subseteq E$ containing both p and q.

2.5:3. *When a union of sets is connected.* (**d**: 2)

(a) Suppose A and B are connected nonempty subsets of a metric space X. Show that $A \cup B$ is connected if and only if A and B are not separated.

(b) Suppose A and B are subsets of a metric space X, and neither A nor B is connected. Can $A \cup B$ be connected?

2.5:4. *Ultrametric spaces are disconnected.* (**d**: 2,3)

(a) Show that a metric space satisfies the *ultrametric inequality* (see exercise **2.4:3** above) if and only if for every three points $p, q, r \in X$, at least two of the distances $d(p,q)$, $d(q,r)$, $d(r,p)$ are equal, and the third is $\leq$ their common value.

(b) Show that a metric space of more than one point which satisfies the ultrametric inequality cannot be connected.

Remark: Using part (a) of **2.4:3** and the above result, one can easily deduce that no subset of the Cantor set having more than one point is connected with respect to the metric d'. However, the property of being connected can be expressed in terms of open sets; hence using (b) of **2.4:3** we can conclude that no subset of the Cantor set having more than one point is connected as a subset of R. Of course, this also

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Answers to True/False question **2.4:0.** (a) F. (b) F.

follows from the observations on p.42 of Rudin.

2.5:5. *Connectedness of compact sets*. (**d**: 3)

Show that a compact metric space X is connected if and only if it cannot be written as a union $X = A \cup B$ with $\inf_{a \in A,\, b \in B} d(a,b) > 0$. Of the two directions in this double implication, you should prove one for arbitrary metric spaces X; only the other direction requires compactness.

2.6. Separable metric spaces (developed only in exercises). (p.45)

Relevant exercises in Rudin:

In the group of exercises given below, most refer to the definition of ‘‘separable metric space’’ given in **2:R22**, and several refer to the concept of a ‘‘base’’ of a metric space defined in **2:R23**. But aside from needing to look at an earlier exercise for a definition, you do not need the results of these exercises unless this is indicated in the dependence statements below.

Rudin will call on the result of **2:R25** in Chapter 7 when proving Theorem 7.25 (though in fact, the preceding paragraph of that proof contains most of the proof of that exercise, so that the reference is not really needed).

2:R22. *R^k is separable*. (**d**: 2)

For ‘‘countable dense subset’’ read ‘‘dense subset which is at most countable’’.

2:R23. *Separability implies the countable base property*. (**d**: 2)

For ‘‘*countable* base’’ read ‘‘base which is at most countable’’.

Rudin really should have combined the result of this exercise with the converse statement; indeed, he seems to assume that converse in **2:R25** when he says ‘‘therefore’’. I give that converse as **2.6:1** below.

2:R24. *Separability and existence of limit points*. (**d**: 3)

2:R25. *Compact metric spaces are separable*. (**d**: 2)

For ‘‘countable base’’ read ‘‘base which is at most countable’’.

If you have done **2:R24**, that can be used in an alternative to the proof that Rudin suggests for this exercise. Even if you haven’t, you might look at the end of the hint to that exercise, to get an idea how the hint to this exercise is to be used.

To get the final statement ‘‘and is therefore separable’’, you should include a proof of **2.6:1** below, if you haven’t done it.

2:R26. *‘‘Infinite sets have limit points’’ implies compactness: a converse to Theorem 2.37*. (**d**: 3. > **2:R23**, **2:R24**)

In the *Hint*, for ‘‘countable base’’ read ‘‘base which is at most countable’’.

An alternative proof of this result will be given as **3.3:4** below.

2:R27. *Condensation points*. (**d**: 3. > **2:R22**, **2:R23**)

You should prove this for an arbitrary separable metric space, then use **2:R22** to get it for R^k.

2:R28. *In a separable metric space, every closed set = perfect $\cup$ countable*. (**d**: 1. > **2:R27**)

If you haven’t proved **2:R27** in the generalized form suggested above, then in Rudin’s Hint, for ‘‘Use’’ read ‘‘Generalize the suggested proof of’’. (And if you haven’t done **2:R27** at all, raise the ‘‘difficulty rating’’ of this one to 3.)

2:R29. *Description of open sets in R*. (**d**: 3. > **2:R22**)

Though only **2:R22** needs to be called on for this one, **2:R23** and its hint can be helpful for seeing the idea to be used.

Exercises not in Rudin:

2.6:1. *The countable basis property implies separability*. (**d**: 2)

Show that every metric space with a countable basis is separable i.e., the converse to **2:R23**. (Hint: Chose one point from each member of such a basis.)

2.6:2. *Separability is inherited by subsets.* (**d**:3)

Show that every subset E of a separable metric space X is separable as a metric space.

(One approach: If S is an at most countable dense subset of X, show that you can choose an at most countable subset T of E such that for every n and i, if E contains a point of whose distance from x_n is $<1/i$, then so does T, and that such a T will be dense in E. Alternative approach: If you have done **2.6:1**, deduce this result from that one. In this case, the difficulty of the problem goes down to **d**:2.)

Chapter 3. Numerical Sequences and Series.

3.1. CONVERGENT SEQUENCES. (pp.47-51)

Relevant exercises in Rudin:

3:R1. *Convergence of* (s_n) *versus* $(|s_n|)$. (**d**:1)

3:R2. $\lim\,(\sqrt{n^2+n} - n)$. (**d**:2)

In this problem you can use the ''trick'' for simplifying such limits from first-year calculus; ask a friend if you didn't learn such a trick. Unfortunately, after the first simplification, the ''obvious'' next step is really an application of continuity of the square root function, and we can't talk about continuity until Chapter 4. So instead, show that the square root in your expression lies between two integers (i.e., between the square roots of two perfect squares), use this to get upper and lower bounds on that expression, and deduce the limit from these bounds.

3:R3. $\lim \sqrt{2+\sqrt{\ldots}}$. (**d**:3)

If you don't see how to begin this one, try computing to a couple of decimal places the first few s_n.

3:R16. *A fast-converging algorithm for square roots.* (**d**:3)

In the first line of this exercise, the words ''fix'', ''choose'', and ''define'' are not instructions to you; Rudin means ''Suppose a positive number α has been fixed'', etc..

3:R17. *Another algorithm for square roots.* (**d**:3)

I haven't marked this exercise as depending on **3:R16** because the only dependence is in part (d), where Rudin asks you to compare the behavior of this algorithm with that of the latter exercise. Interpret this as referring to part (b) of that exercise, and assume that result even if you have not done that exercise.

3:R18. *What does* this *algorithm do?* (**d**:3. >**3:R16**)

Exercises not in Rudin:

3.1:0. *Say whether each of the following statements is true or false.*

(a) If E is a subset of a metric space X, then any sequence of points of E that converges in X converges in E.

(b) If (s_n) and (t_n) are sequences of real numbers such that the sequence (s_n+t_n) is convergent, then the sequences (s_n) and (t_n) are both convergent.

(c) If (s_n) and (t_n) are sequences of real numbers such that the sequences (s_n) and (t_n) are both convergent, then the sequence (s_n+t_n) is convergent.

(d) If (s_n) and (t_n) are sequences of real numbers such that the sequences (s_n) and (s_n+t_n) are both convergent, then the sequence (t_n) is convergent.

(e) Every convergent sequence in R is bounded.

(f) If (s_n) is a convergent sequence in R, and c is a constant such that for all n we have $s_n < c$, then $\lim_{n\to\infty} s_n < c$.

(g) If (s_n) is a convergent sequence in R, and c is a constant such that for all n we have $s_n \leq c$, then $\lim_{n\to\infty} s_n \leq c$.

3.1:1. *A convergent sequence together with the point it approaches form a compact set.* (**d**: 2)

Let X be a metric space.

(a) Show that if (p_i) is a sequence in X which converges to a point p, then the set $\{p\} \cup \{p_i \mid i = 1, 2, ...\}$ is compact.

(b) Deduce that a subset $S \subseteq X$ is closed if and only if for every compact subset $E \subseteq X$, the intersection $S \cap E$ is also compact.

3.2. SUBSEQUENCES. (pp.51-52)

Relevant exercises in Rudin: None

Exercises not in Rudin:

3.2:1. *The converse to Theorem 3.6(a).* (**d**: 1. >**2:R26**)

Deduce from **2:R26** (p.45) the converse to Theorem 3.6(a) (p.51), namely that if X is a metric space such that every sequence in X has a convergent subsequence, then X is compact.

3.2:2. *Sequences in R with prescribed subsequential limit-sets.* (**d**: 3)

Find four sequences in R, whose sets of subsequential limit points are respectively (a) the empty set, (b) the set of integers, (c) the interval $[0, 1]$, and (d) all of R.

(To ''find'' such sequences, you may either give them explicitly, or describe precisely how they may be constructed, possibly in terms of something else that has been proved to exist. You must *prove* the asserted properties of the sequences you have constructed, unless they are quite obvious.)

3.2:3. *The subsequential limit-set of a product sequence $(s_i t_i)$.* (**d**: 2,2,2,3,4)

(a) Let (s_i) and (t_i) be bounded sequences of real numbers, and let E, F be the sets of all subsequential limit points of (s_i) and (t_i) respectively. (Recall that by definition, these are subsets of R. Rudin does not count $\pm\infty$ as subsequential limit points, even if a sequence has $+\infty$ as its lim sup or $-\infty$ as its lim inf.) Show that the set of subsequential limit points of $(s_i t_i)$ (i.e., of the sequence $(s_1 t_1, s_2 t_2, ...))$ is contained in the set $EF = \{ef \mid e{\in}E, f{\in}F\}$.

(b) Show that the inclusion of part (a) can be proper.

(c) Show that the statement of part (a) can fail if (s_i) and (t_i) are not assumed bounded. (Recall that questions like (b) and (c) can only be answered by examples.)

(d) Let (s_i) be a bounded sequence of real numbers and E the set of subsequential limit points of (s_i). Show that the set of subsequential limit points of (s_i^2) is precisely $\{e^2 \mid e{\in}E\}$.

(e) Let $(\mathbf{s}_i)$ be a sequence of points of R^k (or if you are more comfortable with something you can picture, R^2), and let E again be the set of subsequential limit points of $(\mathbf{s}_i)$. Show that the set of subsequential limit points of the sequence of real numbers $(|\mathbf{s}_i|)$ is precisely $\{|\mathbf{e}| \mid \mathbf{e}{\in}E\}$.

3.2:4. *Subsequential limits of unions of sequences.* (**d**: 2,2,3)

Let us call a sequence (a_k) the ''union'' of two subsequences (a_{m_k}) and (a_{n_k}) (where $m_1 < m_2 < ...$ and $n_1 < n_2 < ...$) if $\{m_1, m_2, ...\} \cup \{n_1, n_2, ...\} = \{1, 2, 3, ...\}$.

(a) Suppose a sequence (a_k) in a metric space X is written as the union of two subsequences (a_{m_k}) and (a_{n_k}). Let E, E_1 and E_2 denote the sets of subsequential limit points of (a_k), (a_{m_k}) and (a_{n_k}) respectively. Prove that $E = E_1 \cup E_2$.

Deduce that if $X = R$, then $\limsup_{k\to\infty} a_k = \max(\limsup_{k\to\infty} a_{m_k}, \limsup_{k\to\infty} a_{n_k})$

One can more generally speak of writing a sequence (a_k) as the union of a family of subsequences $(a_{n_{\alpha,k}})$ where α ranges over any index-set A, and get

(b) Prove (or deduce from part (a)) that if a sequence (a_k) in a metric space X is written as the union of finitely many subsequences $(a_{n_{1,k}}), \ldots, (a_{n_{r,k}})$, and we write E for the set of subsequential limit

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Answers to True/False question **3.1:0.** (a) F. (b) F. (c) T. (d) T. (e) T. (f) F. (g) T.

points of (a_k), and for each $j \in \{1, \ldots, r\}$ we write E_j for the set of subsequential limit points of $(a_{n_{j,k}})$, then $E = E_1 \cup \ldots \cup E_r$. Conclude that if $X = R$, then

$$\limsup_{k\to\infty} a_k \;=\; \max(\limsup_{k\to\infty} a_{n_{1,k}}, \ldots, \limsup_{k\to\infty} a_{n_{r,k}}).$$

The next part shows that things are very different for infinite unions of subsequences:

(c) Let (a_k) be a sequence in a metric space, let E be its set of subsequential limit points, and let x be any point of E. Show that (a_k) can be written as the union of countably many subsequences $(a_{n_{1,k}})$, $(a_{n_{2,k}})$, ... such that each subsequence $(a_{n_{i,k}})$ converges to x. Moreover, show that we can take these subsequences to be ‘‘disjoint’’, in the sense that for $i \neq j$, $\{n_{i,1}, n_{i,2}, \ldots\} \cap \{n_{j,1}, n_{j,2}, \ldots\} = \emptyset$.

(Hint: Choose a subsequence (a_{n_k}) of (a_k) that converges to x. Define the subsequences $(a_{n_{i,k}})$ so that each one contains infinitely many terms from (a_{n_k}) and at most one term not belonging to it.)

Deduce that if $X = R$, then $\limsup_{k\to\infty} a_k$ is not determined by the numbers $\limsup_{k\to\infty} a_{n_{j,k}}$.

3.2:5. [*A revised version of the exercise previously having this number is now* **7.1:3**.]

3.2:6. *Some properties of subsequential limit sets*. (**d**: 3, 3, 4, 5, 5, 4, 5)

In (a)-(f) below, let X be a metric space, (s_n) a sequence in X such that $\lim_{n\to\infty} d(s_n, s_{n+1}) = 0$, and E the subsequential limit set of (s_n).

(a) Show that if $X = R$, then E is connected.

(b) Show that for every $\varepsilon > 0$, all but finitely many natural numbers n have the property that there exists $p \in E$ with $d(s_n, e) < \varepsilon$.

(c) Show that if X is compact, then E is connected.

(d) Give an example where $X = R^2$ and E is not connected. (By Rudin’s definition, the empty set is connected, so your example must have $E \neq \emptyset$.)

(e) Show that if $X = R^k$ and E is not connected, then it contains an unbounded connected subset.

(f) Show that if $X = R^k$ and (s_n) is not convergent, then E is perfect.

(g) Show that if X is a connected compact metric space, then there exists a sequence (s_n) with $\lim_{n\to\infty} d(s_n, s_{n+1}) = 0$ whose subsequential limit set is all of X.

3.3. CAUCHY SEQUENCES. (pp.52-55)

Relevant exercises in Rudin:

3:R20. *A Cauchy sequence with a convergent subsequence converges*. (**d**: 1)

If we call the exercise as in Rudin ‘‘part (a)’’, we can add:

(b) From part (a) above and Theorem 3.6, prove Theorem 3.11(b, c) without using Theorem 3.10.

3:R21. *A shrinking sequence of closed sets in a complete metric space has nonempty intersection*. (**d**: 2)

3:R22. *Baire’s Theorem on intersections of dense open subsets*. (**d**: 2. > **3:R21**)

3:R23. *Distances between points of two Cauchy sequences*. (**d**: 2)

3:R24. *The completion of a metric space*. (**d**: 4)

The exercise assumes familiarity with the concept of the set of equivalence classes of an equivalence relation.

3:R25. *What is the completion of the metric space Q?* (**d**: 4. > **3:R24**)

Exercises not in Rudin:

3.3:0. *Say whether each of the following statements is true or false*.

(a) If E is a subset of a metric space X, then any sequence of points of E that is a Cauchy sequence in X is a Cauchy sequence in E.

(b) Every convergent sequence in a metric space is a Cauchy sequence.

(c) Every subset of R^k that is complete as a metric space is bounded.

3.3:1. *Closed subsets and complete subsets*. (**d**: 2,1,4)

The three parts of this exercise prove related facts, but they are independent of one another.

(a) Show that if X is a metric space, and Y a subset of X which is complete as a metric space, then Y is closed in X. Show that both Theorem 2.34 and the result of **2.3:6** above follow from this.

(b) Show that if X is complete as a metric space and Y is closed in X, then Y is also complete.

(c) Show that if Y is a metric space such that for every metric space X containing Y, the set Y is closed in X, then Y is complete. (Suggestion: Prove the result in contrapositive form. If Y is not complete, take an element $q \notin Y$ and find a way to make $X \cup \{q\}$ a metric space in which q is a limit point of X.)

(Remark on (c): Given any metric space Y, there in fact exist *complete* metric spaces X containing Y. If we had that fact available here, we could deduce (c) quickly from (b). Rudin shows two very different ways of constructing such spaces in **3:R24** and **7:R24**. But the former requires familiarity with the concept of the set of equivalence classes of an equivalence relations, while the latter uses material that comes much later in this course.)

3.3:2. *A complete bounded* (*connected*) *metric space need not be compact*. (**d**: 1,4)

(a) Show that the metric space described in **2:R10** (p.44) is bounded and complete, but not compact.

(b) Find an example of a *connected* metric space that is bounded and complete but not compact.

3.3:3. *Sequential test for Cauchy sequences?* (**d**: 1,5)

(a) Show by example that for (s_n) a sequence of points in a metric space, the condition that (s_n) be a Cauchy sequence is *not* equivalent to the condition $\lim_{n\to\infty} d(s_n, s_{n+1}) = 0$. First note why one implication is true; your example should show the other is false.

(b) Does there exist a sequence of pairs of positive integers $((m_k, n_k))$ such that for every sequence of points (s_n) in any metric space, the sequence (s_n) is Cauchy if and only if $\lim_{k\to\infty} d(s_{m_k}, s_{n_k}) = 0$?

Suggestion on (b): Try different sequences of pairs. For some of them you will probably find that a sequence (s_n) can satisfy $\lim_{k\to\infty} d(s_{m_k}, s_{n_k}) = 0$ without being Cauchy, and for others that a Cauchy sequence can fail to have $\lim_{k\to\infty} d(s_{m_k}, s_{n_k}) = 0$. Try to determine what properties of $((m_k, n_k))$ lead to the one or the other difficulty, and use these observations either as a guide in constructing a sequence of pairs having neither difficulty, or to prove that every sequence of pairs must suffer from one or the other.

3.3:4. *The converse to Theorem 3.6*(*a*). (**d**: 2)

Let X be a metric space in which every sequence has a convergent subsequence. You will show below that X is compact.

(This is equivalent to **2:R26**, but we will get the result without the machinery of **2:R22-24** used there.)

Suppose $\{G_\alpha\}$ is an open cover of X.

(a) Let $f: X \to R$ be the function that associates to every point x the supremum of all real numbers $\varepsilon \leq 1$ such that some G_α in our cover contains $N_\varepsilon(x)$. Indicate why the set whose supremum defines $f(x)$ is nonempty and bounded above (so that f is defined), and show that the function f so defined is continuous, and everywhere positive-valued.

(b) Show that $\inf_x f(x) > 0$. (Hint: If it were 0, show that you could find a sequence of points x_n such that $\lim_{n\to\infty} f(x_n) = 0$. Then apply the assumption about convergent subsequences to get a contradiction.)

To complete the proof, let c be a positive real number less than $\inf_x f(x) > 0$, and suppose we select successively, as long as we can, points $x_1, x_2, \ldots \in X$, and members $G_{\alpha_1}, G_{\alpha_2}, \ldots$ of our cover with $N_c(x_i) \subseteq G_{\alpha_i}$, such that $x_i \notin \bigcup_{j<i} G_{\alpha_j}$.

Answers to True/False question **3.3:0.** (a) T. (b) T. (c) F.

(c) Show that if $\{G_\alpha\}$ does not have a finite subcover, we can continue this process indefinitely, getting, in particular, a sequence of points (x_n) such that any two points of this sequence are at distance at least c apart. Show that such a sequence cannot have a convergent subsequence, contradicting our hypothesis.

The above contradiction shows that $\{G_\alpha\}$ must have a finite subcover, hence that X is indeed compact.

3.4. UPPER AND LOWER LIMITS. (pp.55-57)

Relevant exercises in Rudin:

3:R4. *The upper and lower limits of a particular sequence.* (**d**:1)

3:R5. *The upper limit of a sum of sequences.* (**d**:2)

If we call the exercise as Rudin gives it part ''(a)'', we can add

(b) Show by example that the inequality of part (a) can be strict (i.e., that there exist examples in which ''<'' holds).

Exercises not in Rudin:

3.4:0. *Say whether the following statement is true or false.*

(a) If (s_n) is a bounded sequence of real numbers, then there exists a subsequence (s_{n_k}) which converges to $\liminf_{n\to\infty} s_n$.

3.4:1. *Upper and lower limits of a sequence and a subsequence.* (**d**:2)

Suppose (s_n) is a sequence of real numbers, and (s_{n_k}) a subsequence.

(a) Show that if (s_n) converges to the real number s, then so does (s_{n_k}).

(b) Show that (whether or not (s_n) converges), $\limsup_{n\to\infty} s_{n_k} \leq \limsup_{n\to\infty} s_n$, and $\liminf_{n\to\infty} s_{n_k} \geq \liminf_{n\to\infty} s_n$.

(c) Give examples where the inequalities of (b) are strict (i.e., where equality does not hold).

3.4:2. *Interpreting* lim sup *as* lim(sup). (**d**:2)

The symbol ''lim sup'' is an abbreviation for ''limit superior'', i.e., ''upper limit''; but here is another way of ''justifying'' the symbol: If (a_n) is a sequence of real numbers, show that

$$\limsup a_n \;=\; \lim_{m\to\infty} (\sup_{n\geq m} a_n).$$

(The ''sup''s on the right hand side may be extended reals rather than real numbers. Rudin has not defined the limit of a sequence of extended real numbers; so either prove this result under the assumption that the suprema on the right are finite, or state explicitly what happens if one or more of them is $+\infty$ or $-\infty$.)

3.4:3. *Arithmetic of possibly infinite limits.* (**d**:2)

Definition 3.15 (p.55) says what it means for a sequence of real numbers to approach $+\infty$ or $-\infty$. This exercise will extend to that situation basic results proved by Rudin about arithmetic operations on limits of sequences (Theorem 3.3, p.49). The next exercise will do the same for the existence of convergent subsequences (Theorem 3.6(b), p.51).

Suppose first that (x_n), (y_n) are sequences of real numbers, and a, b are extended real numbers such that $x_n \to a$ and $y_n \to b$. (Thus, if a, respectively b, is real, this means convergence in the ordinary sense, while if it is infinite, this means convergence in the sense of Definition 3.15.) We wish to show that if the sum $a+b$ is defined, then $x_n + y_n \to a+b$, and that, similarly, if the product ab is defined, then $x_n y_n \to ab$.

(a) Verify the statement on sums in the three cases (a_1) a and b both finite, (a_2) a finite, $b = +\infty$, and (a_3) $a = b = +\infty$. (For one of these cases you merely need to point to a result proved by Rudin.)

(b) Sketch an argument showing that all cases where $a+b$ is defined can be deduced from these three, using commutativity of addition, and changes of sign. (Regarding what operations on extended real numbers are defined, note the comment for p.12 of Rudin on the errata/addenda sheets.)

(c) Verify the statement on products in the three cases (c_1) a and b both finite, (c_2) a finite and positive, $b = +\infty$, and (c_3) $a = b = +\infty$. (Again, one case you can quote from Rudin.)

(d) Sketch an argument showing that all cases where $a\,b$ is defined can be deduced from these three.

Similar results are true for quotients a/b. I omit these for brevity.

(e) Show, conversely, that in every case where $a+b$, respectively ab is undefined, there exist sequences of real numbers (x_n) and (y_n) such that $x_n \to a$ and $y_n \to b$, but such that $x_n + y_n$, respectively $x_n\,y_n$ does not approach any extended real number. Again, I recommend first doing some ''key'' cases, and then getting the remaining cases from these.

(f) Also give an example, for some pair of extended real numbers a and b such that $a+b$ is not defined, of sequences of real numbers (x_n), (y_n), (x'_n), (y'_n) such that $x_n \to a$, $y_n \to b$, $x'_n \to a$, $y'_n \to b$, but such that $x_n + y_n$ and $x'_n + y'_n$ approach *different* extended real numbers.

(Similar examples exist for all cases where $a+b$ or ab is undefined, but the preceding parts give enough exercise in passing from key cases to general cases.)

3.4:4. *Every real sequence has a subsequence that approaches some extended real.* (**d**: 1)

Show that every sequence of real numbers has either a convergent subsequence, or a subsequence which $\to +\infty$, or a subsequence which $\to -\infty$.

3.4:5. *An infinite cube as k-dimensional analog of the extended real line.* (**d**: 1,3. >**3.4:3**,**3.4:4**)

As Rudin shows us, it is often useful to regard the real line R as a subset of the extended real line $R \cup \{-\infty, +\infty\}$. However, there is no one natural analog of this construction for Euclidean space R^k. This exercise and the next discuss *two* distinct extensions of R^k.

The one we shall consider in this exercise is the set $(R \cup \{-\infty, +\infty\})^k$ of all k-tuples of extended real numbers; i.e., strings $\mathbf{a} = (a_1, \ldots, a_k)$ in which each a_i is either a real number or $-\infty$ or $+\infty$. Given a sequence $(\mathbf{x}_n)$ of elements of R^k and a point $\mathbf{a} \in (R \cup \{-\infty, +\infty\})^k$, let us write $\mathbf{x}_n \to \mathbf{a}$ if for each $i \in \{1,, \ldots, k\}$ one has $x_{n,i} \to a_i$, where $x_{n,i}$ denotes the ith component of $\mathbf{x}_n$. (Note that like Rudin's limits in the extended real line, this concept of limit has *not* been defined in terms of a metric on our set.)

(a) Show that every sequence $(\mathbf{x}_n)$ of elements of R^k has a subsequence which approaches some point of $(R \cup \{-\infty, +\infty\})^k$. (Hint: Apply **3.4:4** to first coordinates, getting a subsequence of $(\mathbf{x}_n)$, then to second coordinates of the terms of that subsequence, etc..)

(b) Given two points $\mathbf{a}$, $\mathbf{b} \in (R \cup \{-\infty, +\infty\})^k$, one would like to define elements $\mathbf{a}+\mathbf{b} \in (R \cup \{-\infty, +\infty\})^k$ and $\mathbf{a}\cdot\mathbf{b} \in R \cup \{-\infty, +\infty\}$ in a way that respects limits of sequences; that is, so that if $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ are sequences in R^k such that $\mathbf{x}_n \to \mathbf{a}$, $\mathbf{y}_n \to \mathbf{b}$, then $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{a}+\mathbf{b}$ and $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{a}\cdot\mathbf{b}$.

Determine for what pairs of elements $\mathbf{a}$ and $\mathbf{b}$ there exist elements ''$\mathbf{a}+\mathbf{b}$'', respectively ''$\mathbf{a}\cdot\mathbf{b}$'', with these properties. Prove in these cases that the desired properties hold, and in all other cases that there exist sequences $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ which approach the indicated points of $(R \cup \{-\infty, +\infty\})^k$, but such that $(\mathbf{x}_n + \mathbf{y}_n)$ does not approach any point of $(R \cup \{-\infty, +\infty\})^k$, respectively such that $(\mathbf{x}_n \cdot \mathbf{y}_n)$ does not approach any element of $R \cup \{-\infty, +\infty\}$. You may assume the result of **3.4:3** above, even if you did not do it.

State also how and under what conditions the product $a\mathbf{b}$ of an extended real number a and an element $\mathbf{b} \in (R \cup \{-\infty, +\infty\})^k$ can be defined so as to respect limits of sequences. (The verification is so close to that of the case of the dot product $\mathbf{a}\cdot\mathbf{b}$ above that I won't ask you to give it.)

3.4:6. *An infinite ball as k-dimensional analog of the extended real line.* (**d**: 2,3,2,4,4. >**3.4:3**,**3.4:4**)

Here is another way of defining ''infinite limits'' of sequences in R^k.

Suppose $\mathbf{a}$ is a point in R^k satisfying $|\mathbf{a}| = 1$, i.e., lying on the sphere of radius 1 centered at 0. If $(\mathbf{x}_n)$ is a sequence in R^k, let us write $\mathbf{x}_n \to \mathbf{a}\infty$ if $|\mathbf{x}_n| \to +\infty$ and $\mathbf{x}_n / |\mathbf{x}_n| \to \mathbf{a}$. (In the latter

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Answer to True/False question **3.4:0.** (a) T.

limit statement we must drop any terms for which $\mathbf{x}_n = 0$; but if $|\mathbf{x}_n| \to +\infty$ there can be only finitely many such n, so this is no real problem.)

(a) Show that every sequence $(\mathbf{x}_n)$ in R^k has a subsequence which either approaches a point of R^k or approaches some $\mathbf{a}\infty$ in the sense defined above. (Hint: Show that $(\mathbf{x}_n/|\mathbf{x}_n|)$ is bounded.)

(b) Show that if for every $\mathbf{a}$ satisfying $|\mathbf{a}| = 1$, we define $\mathbf{a}\infty + \mathbf{a}\infty = \mathbf{a}\infty$, then this operation respects limits of sequences, in the sense that if $\mathbf{x}_n \to \mathbf{a}\infty$ and $\mathbf{y}_n \to \mathbf{a}\infty$, then $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{a}\infty$.

(c) Show similarly that for $\mathbf{a}$ as above and $\mathbf{p} \in R^k$, the definition $\mathbf{p} + \mathbf{a}\infty = \mathbf{a}\infty$ respects limits of sequences.

(d) On the other hand, if $\mathbf{a}, \mathbf{b} \in R^k$ are distinct points satisfying $|\mathbf{a}| = |\mathbf{b}| = 1$, show that one cannot define $\mathbf{a}\infty + \mathbf{b}\infty$ in a way that will respect limits of sequences. Namely, show that for every such pair of points $\mathbf{a}$ and $\mathbf{b}$ there exist sequences $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ satisfying $\mathbf{x}_n \to \mathbf{a}\infty$, $\mathbf{y}_n \to \mathbf{b}\infty$, such that $\mathbf{x}_n + \mathbf{y}_n$ does not approach $\mathbf{c}\infty$ for any $\mathbf{c}$, nor any point of R^k. (Suggestion: Get an example where the odd-subscripted terms $\mathbf{x}_{2n+1} + \mathbf{y}_{2n+1}$ approach $\mathbf{a}\infty$ and the even-subscripted terms $\mathbf{x}_{2n} + \mathbf{y}_{2n}$ approach $\mathbf{b}\infty$.)

(e) Determine similarly the conditions under which the dot product of two ‘‘infinite points’’ $(\mathbf{a}\infty)\cdot(\mathbf{b}\infty)$, and the dot product of a ‘‘finite’’ and an ‘‘infinite’’ point, $(\mathbf{a}\infty)\cdot\mathbf{p}$, can be defined in a manner that respects limits of sequences.

3.4:7. *Convergence in the sense of* **3.4:5** *and in the sense of* **3.4:6** *are almost independent*. (**d**: 4,4,2. >**3.4:5**,**3.4:6**)

(a) Show by examples that a divergent sequence in R^k can approach a point of $(R \cup \{-\infty, +\infty\})^k$ in the sense of **3.4:5** above, but not approach any $\mathbf{a}\infty$ in the sense of **3.4:6**, and similarly that such a sequence can approach a point $\mathbf{a}\infty$ in the sense of the latter exercise, but not approach any point of $(R \cup \{-\infty, +\infty\})^k$ in the sense of the former exercise.

(b) Which points $\mathbf{p}$ of $(R \cup \{-\infty, +\infty\})^k$ have the property that every sequence in R^k which approaches $\mathbf{p}$ in the sense of **3.4:5** approaches a point $\mathbf{a}\infty$ in the sense of **3.4:6**? Which points $\mathbf{a}\infty$ have the property that every sequence in R^k which approaches $\mathbf{a}\infty$ in the sense of **3.4:5** approaches a point $(R \cup \{-\infty, +\infty\})^k$ in the sense of **3.4:6**?

(c) Show that every unbounded sequence in R^k has a *subsequence* which both approaches a point of $(R \cup \{-\infty, +\infty\})^k$ in the sense of the former exercise and approaches a point $\mathbf{a}\infty$ in the sense of the latter.

3.4:8. *Incomparable sets of natural numbers*. (**d**: 2,5)

This exercise has little to do directly with Real Analysis, except for getting one thinking about the operation ‘‘lim sup’’; but I find it interesting.

For every set S of natural numbers and every natural number n, let

$$\mathrm{in}(S,n) = \text{the number of natural numbers } m < n \text{ such that } m \in S,$$
$$\mathrm{out}(S,n) = \text{the number of natural numbers } m < n \text{ such that } m \notin S.$$

So, $\mathrm{out}(S,n) = n - \mathrm{in}(S,n)$.

(a) Show that if T is a subset of the natural numbers and S is a proper subset of T, then $\limsup_{n\to\infty}(\mathrm{in}(S,n) - \mathrm{out}(S,n)) < \limsup_{n\to\infty}(\mathrm{in}(T,n) - \mathrm{out}(T,n))$, unless both of these are $+\infty$ or both are $-\infty$.

Now let A be the set of all subsets of the natural numbers such that

$$\limsup_{n\to\infty}(\mathrm{in}(S,n) - \mathrm{out}(S,n)) = 0.$$

Thus, we see from (a) that no member of A is a proper subset of any other member of A.

(b) Show that every set S of natural numbers either contains (as a subset) some member of A, or is contained (as a subset) in some member of A.

3.5. SOME SPECIAL SEQUENCES. (pp.57-58)

Relevant exercises in Rudin: None

Exercises not in Rudin:

3.5:0. *Say whether each of the following statements is true or false.*

(a) $\lim_{n\to\infty} n^{1/n} = 1.$

(b) $\lim_{n\to\infty} n^{1{,}000}(1.001)^{-n} = +\infty.$

3.5:1. $\lim_{n\to\infty} n^{n^{-p}}$. (**d**: 4)

Prove or disprove: For every positive real number p, one has $\lim_{n\to\infty} n^{n^{-p}} = 1$. (Rudin proves the case $p = 1$.)

3.6. SERIES. (pp.58-61)

Relevant exercises in Rudin:

3:R6. *Test some series for convergence.* (**d**: 2)

If you don't see how to do (*a*), start by writing out the first few partial sums.

3:R14. *Convergence in the mean.* (**d**: 2,2,3,3,4)

3:R15. *Extending the results of this chapter to* R^k. (**d**: ?)

This exercise asks you to show that the main results on summation of series in R in this chapter are also true in R^k, with ''only very slight modifications'' in the proofs. I have placed it here because the first three of the nine theorems listed in the exercise are from this section; the remaining six belong to later sections. I am not sure what it would mean to assign this exercise as homework; there would have to be clear instructions on how the student is to present the ''very slight modifications''. Perhaps this exercise should merely be looked at as a guide to the student interested in thinking about the subject.

3:R19. *The Cantor set as the set of sums of certain series.* (**d**: 3)

Exercises not in Rudin:

3.6:0. *Say whether each of the following statements is true or false.*

(a) If $0 \le a_n \le 3^{-n}$ for all but finitely many n, then $\Sigma_{n=1}^{\infty} a_n$ converges.

(b) If (a_n) is a sequence of real numbers such that the set of partial sums $\{\Sigma_{k=1}^{n} a_k \mid n \in J\}$ is bounded, then $\Sigma_{n=1}^{\infty} a_n$ converges.

3.6:1. *Does the analog of Theorem 3.24 hold for these regions in* R^2*?* (**d**: 2)

Theorem 3.24 (p.60) concerns convergence of series of nonnegative real numbers. If we want to generalize it to series in R^2, we need to decide what set we will use in place of the set of nonnegative reals. Consider the following three sets. (Suggestion: draw pictures.)

$$E_1 = \{(x,y) \mid x \ge 0,\ y \ge 0\},$$
$$E_2 = \{(x,y) \mid x \ge 0,\ x+y \ge 0\},$$
$$E_3 = \{(x,y) \mid x+|y| \ge 0\}.$$

For each of these sets, determine whether the statement obtained from Theorem 3.24 by replacing ''nonnegative terms'' with elements of that set is true or false. (As usual, you must prove your answers.)

3.6:2. *Multiplicative series* $\Pi\, a_n$. (**d**: 4)

Let (a_n) be a sequence of real numbers. Then $\Pi_{n=p}^{q} a_n$ denotes the product $a_p a_{p+1} \ldots a_q$, and one defines $\Pi_{n=1}^{\infty} a_n = \lim_{q\to\infty} \Pi_{n=1}^{q} a_n$ if this limit exists. One says that this infinite product ''converges'' if the above limit exists *and is nonzero*.

(a) Suppose that either all a_n are > 1, or all are < 1. Show that the infinite product $\Pi_{n=1}^{\infty} a_n$ converges if and only if the infinite sum $\Sigma_{n=1}^{\infty} (a_n - 1)$ converges.

(b) Show by example that the result of part (a) fails if we do not assume that either all $a_n > 1$ or all

$a_n < 1$. (Suggestion: Let the sequence have the form $1+c_1,\ 1-c_1,\ 1+c_2,\ 1-c_2, \ldots,\ 1+c_n,\ 1-c_n, \ldots$ where $c_n \in (0,1)$, and examine conditions for the above sum and product respectively to converge.)

3.6:3. *The Cantor set, and the* 2*-adic metric on* Z. (**d**: 4. >**2.2:14**, **2.4:3**, **3:R19**)

The result of Rudin's exercise **3:R19** is equivalent to saying that the Cantor set consists of precisely those real numbers which can be written in base 3 using only 0's and 2's to the right of the decimal point, and nothing to the left. (For instance, the largest element of the Cantor set, namely 1, can be written .2222... in base 3, just as it can be written .9999.... in base 10.) Now consider the following function f from the nonnegative integers to the Cantor set:

Given a nonnegative integer n, write n in base 2 (binary) notation, then reverse the order of digits, precede the resulting symbol by a decimal point, change all the ''1''s to ''2''s, and regard the result as an expression for a real number $f(n)$ in base 3. For example, taking $n = 6$, which in binary notation is 110, our construction gives the number whose base-3 expression is .022. This is $8/27$, so $f(6) = 8/27$.

Show that for positive integers m and n, one has $d_2(m,n) = d(f(m), f(n))$, where d_2 is defined as in **2.2:14**, and d as in **2.4:3**. Show also that $f(J)$ is dense in the Cantor set.

3.6:4. *The space* ℓ^2 *of square-summable sequences.* (**d**: 2, 2, 1, 3)

Let ℓ^2 (pronounced ''little-ell-2'') denote the set of all sequences $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$ of real numbers such that

$$\Sigma_{n=1}^{\infty} |x_n|^2 \text{ converges.}$$

Such a sequence is called *square-summable*. (I used absolute-value signs above to put the definition in a form applicable to complex sequences as well, but we will only consider real sequences in this exercise.) For $\mathbf{x} \in \ell^2$, we define

$$|\mathbf{x}| = (\Sigma_{n=1}^{\infty} |x_n|^2)^{1/2}.$$

In this exercise, sums and scalar multiples of sequences of real numbers will be defined by the same rules used for sums and scalar multiples of elements of R^k in Definition 1.36 (p.16). A general technique recommended for most of the steps below is to use Theorem 1.37 (p.16) to estimate partial sums.

(a) Show that if $\mathbf{x}, \mathbf{y} \in \ell^2$ and $\alpha \in R$, then $\alpha\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ are also in ℓ^2, and that the series $\Sigma_{n=1}^{\infty} x_n y_n$ converges. (Its sum is denoted $\mathbf{x}\cdot\mathbf{y}$.)

(b) Show that if we define $d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$, then this definition makes ℓ^2 a metric space.

(c) For each positive integer n, let $\mathbf{e}_n \in \ell^2$ be the sequence whose nth component is 1, while all other components are 0. Show that the sequence $\mathbf{e}_1, \mathbf{e}_2, \ldots$ in ℓ^2 has the property that for each m, the mth coordinates of the terms of the sequence converge to 0, but the sequence itself is not convergent.

(d) Show that ℓ^2 is a complete metric space.

Remark: The space ℓ^2 is an example of what is known as a *Hilbert space*, that is, a *complete inner product space*. (''Complete'' in the sense of this course; ''inner product space'' in the sense of Math 110.) All finite-dimensional inner product spaces are Hilbert spaces; it is the infinite-dimensional ones, such as ℓ^2, that make the subject particularly interesting. Hilbert spaces over both the real and complex numbers are studied; I would have used complex sequences above, except that Rudin only states Theorem 1.37 for tuples of real numbers.

Another construction of a Hilbert space, using square-*integrable* functions rather than square-*summable* sequences (and based on Lebesgue integration, hence not within the scope of Math 104) is studied in the very last section of Rudin (Definition 11.34 on p.326; cf. last sentence before the exercises on p.332). From the results of that section, in particular the formula (107), one can show that the Hilbert space of square-summable sequences and the Hilbert space of square-integrable functions are isomorphic.

3.6:5. *A divergence result.* (**d**: 3)

On p.55, Rudin defines what it means for a sequence of real numbers to satisfy $s_n \to +\infty$ or $s_n \to -\infty$, noting that a sequence which satisfies one of those conditions is still *not* said to converge.

Answers to True/False question **3.5:0.** (a) T. (b) F. **Answers** to True/False question **3.6:0.** (a) T. (b) F.

(Thus, a nonconvergent sequence of real numbers either approaches $+\infty$ or approaches $-\infty$ or does not approach any extended real number.) On p.59, the equation $\Sigma a_n = s$ is defined to mean $s_n \to s$, where (s_n) is the sequence of partial sums. Combining these conventions we have definitions of the conditions $\Sigma a_n = +\infty$ and $\Sigma a_n = -\infty$.

Suppose now that (a_n) and (b_n) are sequences of real numbers satisfying $a_n \leq b_n$ for all n. Show that if Σa_n does not converge, and is not $-\infty$, then Σb_n does not converge.

3.7. SERIES OF NONNEGATIVE TERMS (Convergence by grouping). (pp.61-63)

Relevant exercises in Rudin:

3:R11. *For every convergent series of positive terms, there is a slower-converging series.* (**d**:4)

3:R12. *For every divergent series of positive terms, there is a slower-diverging series.* (**d**:4)

In place of the above two exercises I recommend **3.7.1** below, which obtains the same results in a more natural way.

Exercises not in Rudin:

3.7:0. *Say whether the following statement is true or false.*

(a) $\Sigma_{n=1}^{\infty} 1/(n^{1/2}(\log n)^2)$ converges.

3.7:1. *No ‘‘boundary’’ between convergent and divergent series.* (**d**:1,3,3. 3.7:1)

The statement Rudin makes on p.63 that there is no ‘‘boundary’’ between convergent and divergent series can be made precise as follows:

Given sequences (a_n), (b_n) of positive real numbers, each having limit 0, let us say that ‘‘(a_n) decays more rapidly than (b_n)’’, equivalently that ‘‘(b_n) decays more slowly than (a_n)’’ if $\lim_{n\to\infty} a_n/b_n = 0$, equivalently, $\lim_{n\to\infty} b_n/a_n = +\infty$. We note that if a sequence of positive terms has convergent sum, so does every sequence of positive terms which decays more rapidly; and if a sequence of positive terms has divergent sum, then so does every sequence of positive terms which decays more slowly.

Let us call (a_n) a ‘‘boundary sequence’’ if every sequence that decays more rapidly than (a_n) converges and every sequence that decays more slowly than (a_n) diverges. Rudin’s **3:R11** and **3:R12** show respectively that given a divergent series Σa_n, one can find a sequence (b_n) which decays more rapidly than (a_n) but such that Σb_n still diverges, and that given a convergent series Σa_n, one can find a sequence (b_n) which decays more slowly than (a_n) but such that Σb_n still converges.

(a) Deduce from the above facts that no series Σa_n, whether divergent or convergent, can satisfy the definition given above for a ‘‘boundary sequence’’ (thus justifying Rudin’s claim).

However, the constructions of **3:R11** and **3:R12** are unnecessarily complicated. Here is a simpler pair of constructions:

(b) Suppose Σa_n is a divergent series of positive terms with $\lim_{n\to\infty} a_n = 0$. Show that there is an increasing sequence (A_n) of positive real numbers with $\lim_{n\to\infty} A_n = +\infty$ such that $A_{n+1} - A_n = a_n$ for all n. Now let $B_n = A_n^{1/2}$, and define $b_n = B_{n+1} - B_n$. Show that (b_n) is also a sequence of positive terms that approaches 0, and that it decays more rapidly than (a_n), but that (B_n) still approaches $+\infty$, hence that Σb_n is still divergent.

(Hint: Here and in part (b), the relation $B_n = A_n^{1/2}$ is more easily used in the form $A_n = B_n^2$.)

(c) Suppose Σa_n is a convergent series of positive terms. Show that there is a decreasing sequence (A_n) of positive real numbers with $\lim_{n\to\infty} A_n = 0$ such that $A_n - A_{n+1} = a_n$ for all n. Again let $B_n = A_n^{1/2}$ and now let $b_n = B_n - B_{n+1}$. Show that (B_n) also approaches 0, hence that Σb_n converges, but that (b_n) decays less rapidly than (a_n).

3.7:2. *We can’t test Σa_n for convergence by looking at the terms $a_{2^{2^k}}$ – or can we?* (**d**:3)

(a) Show that there exist two decreasing sequences of positive real numbers, $a_1 \geq a_2 \geq \ldots$ and $b_1 \geq$

$b_2 \geq \ldots$, such that for all k, $a_{2^{2^k}} = b_{2^{2^k}}$, and such that $\Sigma\, a_n$ converges while $\Sigma\, b_n$ does not. (Suggestion: Starting with any decreasing sequence (c_n) of positive real numbers, show that there is a ''smallest'' decreasing sequence (a_n) such that for every k, $a_{2^{2^k}} = c_k$, and a ''largest'' decreasing sequence (b_n) such that for every k, $b_{2^{2^k}} = c_k$. Write down formulas for $\Sigma\, a_n$ and $\Sigma\, b_n$, and look for a case where the former converges but the latter does not.)

(b) Find the fallacy in the following argument: ''If $a_1 \geq a_2 \geq \ldots$ is a sequence of positive real numbers, then by Theorem 3.27 (p.61), $\Sigma\, a_n$ converges if and only if $\Sigma\, 2^k a_{2^k}$ converges, and repeating the argument, this converges if and only if $\Sigma\, 2^k 2^{2^k} a_{2^{2^k}}$ converges. Since this sum depends only on the terms $a_{2^{2^k}}$, it follows that if $b_1 \geq b_2 \geq \ldots$ is another sequence such that for all k, $a_{2^{2^k}} = b_{2^{2^k}}$, then $\Sigma\, a_n$ will converge if and only if $\Sigma\, b_n$ converges.''

3.7:3. *Generalizing the test for convergence by sampling* (*Theorem 3.27*). (**d**:3. >**3.7:2**)

From the preceding exercise, we see that we cannot get a result like Theorem 3.27 (p.61) with the sequence of powers of 2 that is used to ''sample'' the values of a_n replaced by an arbitrary increasing sequence. In this and the next exercise, we shall determine for what increasing sequences such a result does in fact hold.

Suppose $a_1 \geq a_2 \geq \ldots$ is a decreasing sequence of nonnegative real numbers, and $1 = n_1 < n_2 < n_3 < \ldots$ a strictly increasing sequence of integers.

(a) Prove that for all integers $K > 1$,

$$\Sigma_{k=1}^{K-1}\, (n_{k+1} - n_k)\, a_{n_k} \;\geq\; \Sigma_{n=1}^{n_K - 1}\, a_n \;\geq\; \Sigma_{k=1}^{K-1}\, (n_{k+1} - n_k)\, a_{n_{k+1}} \;=\; \Sigma_{k=2}^{K}\, (n_k - n_{k-1})\, a_{n_k}.$$

(b) Deduce that

$$\Big(\Sigma_{k=1}^{\infty}\, (n_{k+1} - n_k)\, a_{n_k} \text{ converges}\Big) \;\Rightarrow\; \Big(\Sigma_{n=1}^{\infty}\, a_n \text{ converges}\Big) \;\Rightarrow\; \Big(\Sigma_{k=2}^{\infty}\, (n_k - n_{k-1})\, a_{n_k} \text{ converges}\Big).$$

(c) Conclude that if the set of ratios $\{(n_{k+1} - n_k)/(n_k - n_{k-1}) \mid k \geq 2\}$ is bounded, then

$$\Big(\Sigma_{n=1}^{\infty}\, a_n \text{ converges}\Big) \;\Leftrightarrow\; \Big(\Sigma_{k=1}^{\infty}\, (n_{k+1} - n_k)\, a_{n_k} \text{ converges}\Big).$$

(d) Deduce from (c) that if $\{(n_{k+1} - n_k)/(n_k - n_{k-1}) \mid k \geq 2\}$ is bounded, and if $b_1 \geq b_2 \geq \ldots$ is another decreasing sequence of positive terms, such that $b_{n_k} = a_{n_k}$ for all k, then $\Sigma_{n=1}^{\infty}\, b_n$ converges if and only if $\Sigma_{n=1}^{\infty}\, a_n$ does.

Exercise **3.7:2** is an example where the above set of ratios is unbounded, and the above equivalence fails.

3.7:4. *A necessary and sufficient condition for convergence to be testable by sampling*. (**d**:1,4. >**3.7:3**)

The preceding exercise found a condition on an increasing sequence of positive integers n_k which is sufficient for the convergence of any decreasing series $\Sigma\, a_n$ of positive terms to be determinable from the ''sampling'' of values a_{n_k}. However, part (a) below shows that this condition is not necessary. Part (b) will establish a condition that is both necessary and sufficient.

(a) Let (n_k) be the increasing sequence of positive integers such that $\{n_k\} = \{2^i \mid i > 0\} \cup \{2^i + 1 \mid i > 0\}$. (Explicitly, $n_{2k-1} = 2^k$, $n_{2k} = 2^k + 1$.) Show that if (a_n) and (b_n) are decreasing sequence of positive terms such that $b_{n_k} = a_{n_k}$ for all k, then $\Sigma_{n=1}^{\infty}\, b_n$ converges if and only if $\Sigma_{n=1}^{\infty}\, a_n$ does. (Hint: The above sequence of integers has a subsequence to which the preceding exercise is applicable.)

The above example suggests that we might incorporate the existence of appropriate subsequences into our criterion. And in fact, doing so gives the desired necessary and sufficient condition:

(b) Show that for positive integers $1 \leq n_1 < n_2 < n_3 < \ldots$, the following three conditions are equivalent:

(i) For every decreasing sequence of positive terms (a_n), the series $\Sigma\, a_n$ converges if and only if the series $\Sigma\, (n_{k+1} - n_k)\, a_{n_k}$ converges.

Answer to True/False question **3.7:0.** (a) F.

(ii) If (a_n) and (b_n) are decreasing sequence of positive terms such that $b_{n_k} = a_{n_k}$ for all k, then the series $\Sigma\, a_n$ converges if and only if $\Sigma\, b_n$ does.

(iii) The set of ratios n_{k+1}/n_k is bounded.

(iv) (n_k) has a subsequence (n_{m_k}) such that the set of ratios $(n_{m_{k+1}} - n_{m_k})/(n_{m_k} - n_{m_{k-1}})$ is bounded.

Suggestion: Prove (i)⇒(ii)⇒(iii)⇒(iv)⇒(i). The second implication is the hardest; I suggest proving it in contrapositive form, using the idea of **3.7:2**. For the third implication, to get the subsequence (n_{m_k}), suppose m_k has been chosen; then let m_{k+1} be the least integer such that $n_{m_{k+1}} \geq 2\, n_{m_k}$.

3.7:5. *Series that* can *be tested for convergence by looking at the terms* $a_{2^{2^k}}$. (**d**: 2. >**3.7:2**)

The fallacious argument discussed in **3.7:2**(b) can in fact be made to work, if we just add an additional condition to our series.

Namely, show that if (a_n) is a sequence of real numbers satisfying the stronger condition $a_1 \geq 2\,a_2 \geq \ldots \geq n\,a_n \geq \ldots$, then $\Sigma\, a_n$ converges if and only if $\Sigma\, 2^k 2^{2^k} a_{2^{2^k}}$ does. Deduce that if (b_n) is another sequence with the same property, and for all k, $a_{2^{2^k}} = b_{2^{2^k}}$, then $\Sigma\, a_n$ converges if and only if $\Sigma\, b_n$ does.

3.7:6. *The series of distances associated with a convergent sequence.* (**d**: 2)

Suppose (p_n) is a sequence in a metric space X which converges to a point p.

(a) Show that $d(p_1, p) \leq \Sigma_{n=1}^{\infty}\, d(p_n, p_{n+1})$.

(b) Show that for every $\varepsilon > 0$ there exists a subsequence (p_{n_k}) with $n_1 = 1$ such that $\Sigma_k\, d(p_{n_k}, p_{n_{k+1}}) < d(p_1, p) + \varepsilon$.

Now suppose (p_n) is an arbitrary sequence of points in X.

(c) Show that if the series $\Sigma_{n=1}^{\infty}\, d(p_n, p_{n+1})$ converges, then (p_n) is a Cauchy sequence; but that the converse is not true. (Hint for the last part: Look for a counterexample in a metric space we are very familiar with.)

3.8. THE NUMBER *e*. (pp.63-65)

Relevant exercises in Rudin: None

Exercises not in Rudin: None

3.9. THE ROOT AND RATIO TESTS. (pp.65-69)

Relevant exercises in Rudin: None

Exercises not in Rudin:

3.9:0. *Say whether each of the following statements is true or false.*

(a) If (a_n) is a sequence such that $\limsup_{n\to\infty} |a_{n+1}/a_n| = ½$, then $\Sigma_{n=1}^{\infty}\, a_n$ converges.

(b) If (a_n) is a sequence such that $\limsup_{n\to\infty} |a_{n+1}/a_n| = 2$, then $\Sigma_{n=1}^{\infty}\, a_n$ diverges.

3.9:1. *Sharper convergence tests analogous to the ratio and root tests.* (**d**: 1, 3, 2, 2, 4, 2)

The idea behind the ratio and root tests is to compare a given series with series of the form $\Sigma\, c x^n$, whose behavior we know for each value of x. But we have seen that series of the form $\Sigma\, n^{-p}$ are too delicate for those tests to work on. Is it possible to devise tests that would similarly compare a series with series of the form $\Sigma\, n^{-p}$?

Yes; such tests are given below. The idea, analogous to the idea behind the ratio and root tests, is to find a limit-formula that, given a sequence of the form $\Sigma\, n^{-p}$, will determine the value of p, and then apply it to more general series.

The analog of the root test, based on considering the ''long-term'' change in magnitude of the terms of the series, is given in part (b), and is relatively easy to state and prove if we again allow ourselves to assume basic properties of the logarithm function. The analog of the ratio test, based on considering the relation between successive terms, is noted in part (e). Like the ratio test, it is generally easier to use than the other criterion, but cannot handle series whose terms don't decrease regularly. It is, unfortunately, hard to prove without the use of differentiation, which we have not yet defined. (The one way I see that one could prove it at this stage is by taking a rational number j/k between p and 1, and using inequalities obtained from the binomial theorem for exponents j and k.) However, I will state both tests here for their interest.

(a) If p is any positive real number, and we let $a_n = n^{-p}$, show that $-\lim_{n\to\infty} (\log a_n)/(\log n) = p$.

(b) Let (a_n) be any sequence of positive terms, and let $p = -\limsup\,(\log a_n)/(\log n)$. Show that if $p > 1$, then $\Sigma\, a_n$ converges.

(c) What happens when we apply this test to a series which the root test shows to converge?

(d) For p again any positive real number and $a_n = n^{-p}$, show that $\lim_{n\to\infty} n(1-(a_{n+1}/a_n)) = p$.

(e) Let (a_n) be any sequence of positive terms, and let $p = \liminf\, n(1-(a_{n+1}/a_n))$. Show that if $p > 1$, then $\Sigma\, a_n$ converges.

(f) What happens when we apply this test to a series which the ratio test shows to converge?

3.9:2. *A slightly modified ratio test.* (**d**: 3,1)

(a) Show that a series $\Sigma\, a_n$ converges if for some positive integer c one has

$$\limsup_{n\to\infty} |a_{n+c}/a_n| < 1.$$

(b) Show that this condition is satisfied for $c = 2$ by both the series of Examples 3.35 (p.67).

3.10. POWER SERIES. (pp.69-70)

Relevant exercises in Rudin:

3:R9. *Finding the radii of convergence of some power series.* (**d**: 2)

3:R10. *The radius of convergence of a power series with integer coefficients.* (**d**: 2)

Exercises not in Rudin:

3.10:0. *Say whether the following statement is true or false.*

(a) If a power series $\Sigma_{n=1}^{\infty} c_n z^n$ converges at $z = 1+3i$, then it also converges at $z = 2+2i$.

3.10:1. *Power series and the ratio test.* (**d**: 1)

Let $\Sigma\, c_n z^n$ be a power series in which all the coefficients c_n are nonzero.

(a) Show that if the sequence $(|c_n/c_{n+1}|)$ approaches either a real number or $+\infty$, then that limit is equal to the radius of convergence of the power series.

(b) If we do not assume that $(|c_n/c_{n+1}|)$ approaches a limit (real or infinite), obtain an inequality relating $\limsup |c_n/c_{n+1}|$ and the radius of convergence of the given power series.

(c) Likewise, obtain an inequality relating $\liminf |c_n/c_{n+1}|$ and the radius of convergence of the power series.

(Hint: You do not have to give any nontrivial arguments to get parts (a)-(c); everything can be obtained easily from results in Rudin.)

(d) Consider the series $\Sigma\, c_n z^n$ where for each nonnegative integer k, we let $c_{2k} = c_{2k+1} = 2^k$. Determine the radius of convergence of this series using Theorem 3.39, determine the upper and lower limits described in (b) and (c) above, and verify for this series the inequalities proved there.

Answers to True/False question **3.9:0.** (a) T. (b) F.

3.11. SUMMATION BY PARTS. (pp.70-71)

Relevant exercises in Rudin:

3:R7. *If* $\Sigma\, a_n$ *converges, so does* $\Sigma\, \sqrt{a_n}/n$. (**d**: 3)

Suggestion: Write $a_n = b_n^2$ and use the Schwarz inequality. Note that the page ends with a very short line which gives one of the hypotheses of the exercise.

3:R8. *Tweaking the hypotheses of Theorem 3.42.* (**d**: 2)

Exercises not in Rudin:

3.11:0. *Say whether each of the following statements is true or false.*

(a) The series $\Sigma\,(-1)^n(1+n^{-1})$ is convergent.

(b) The series $1/1+1/2-2/3+1/4+1/5-2/6+\dots$ is convergent, where the nth denominator is n, and the nth numerator is 1 if n is not divisible by 3 and -2 if n is divisible by 3.

3.11:1. *Symmetrizing the ''summation by parts'' formula.* (**d**: 2)

In display (20) on p.70 of Rudin, the two sums have different ranges of summation. Obtain a similar formula in which the two sums are over the same range of values of n, by adding to the sum on the right-hand side the missing $n = q$ term, subtracting the same term from the end of the formula, and simplifying the result by canceling a pair of terms. (The formula you get should, like (20), have a summation on each side of the equality, and a pair of lone terms.)

3.11:2. *A generalization of* $\Sigma\,(-1)^n n^{-1}$. (**d**: 2)

Let $a_1, a_2, \dots, a_d$ be a finite sequence of complex numbers, and extend it to an infinite sequence by making it periodic of period d, i.e., letting $a_{dk+i} = a_i$ for all positive integers k and all $i\in\{1,\dots,d\}$. Prove that $\Sigma_{n=1}^{\infty}\, a_n/n$ converges if and only if $a_1+a_2+\dots+a_d = 0$.

3.11:3. *A version of Theorem 3.42 with complex* b_n. (**d**: 2)

In Theorem 3.42, p.70, the a_n can be complex numbers, but the b_n are necessarily real, by Rudin's convention that inequality signs such as ''$\geq$'' are only written between real numbers. However, we can replace condition (b) of that theorem with a condition applicable to complex numbers as well:

(a) Show that Theorem 3.42 remains true if condition (b) of that theorem is replaced by the assumption that $\Sigma\,|b_{n+1}-b_n|$ converges.

(b) Show that the assumption of part (a) above does in fact hold whenever conditions (b) and (c) of Theorem 3.42 hold. (Thus, the result proved in part (a) includes that theorem.)

Remark: From the above generalization of Theorem 3.42, we can immediately get the corresponding version of Theorem 3.44. The same method of proof that gives (a) above also easily gives a version where (a_n) and (b_n) are replaced by sequences in R^k, $(\mathbf{a}_n)$ and $(\mathbf{b}_n)$, and the conclusion is that $\Sigma\,\mathbf{a}_n\cdot\mathbf{b}_n$ converges.

3.12. ABSOLUTE CONVERGENCE. (pp.71-72)

Relevant exercises in Rudin: None

Exercises not in Rudin:

3.12:0. *Say whether the following statement is true or false.*

(a) The series $1/1+1/2-2/3+1/4+1/5-2/6+\dots$ (cf. **3.11:0**(b)) is absolutely convergent.

3.12:1. *Achieving absolute convergence by grouping.* (**d**: 1,2)

(a) Suppose $\Sigma_n\, a_n$ is a convergent series. Show that for any sequence of integers $n_1 < n_2 < \dots < n_k < \dots$ with $n_1 = 1$, if for each positive integer k we define

$$A_k \;=\; a_{n_k} + a_{n_k+1} + a_{n_k+2} + \dots + a_{n_{k+1}-2} + a_{n_{k+1}-1},$$

Answer to True/False question **3.10:0.** (a) T.

then the series $\Sigma_k A_k$ converges, and $\Sigma_k A_k = \Sigma_n a_n$.

(b) Show that for any convergent series $\Sigma_n a_n$, there exists a sequence $n_1 < n_2 < \ldots$ as in (a) such that the series $\Sigma_k A_k$ is absolutely convergent. (Hint: For any convergent series Σc_n of positive terms, choose to n_k to make ΣA_k converge by the comparison test with Σc_k.)

3.12:2. *Power series converge absolutely inside their radii of convergence.* (**d**: 1, 2)

Let $\Sigma a_n z^n$ be a power series with radius of convergence $R > 0$

(a) Show that for all complex numbers z with $|z| < R$, the given series converges absolutely.

(b) Show by three examples that for a complex number z with $|z| = R$, the series $\Sigma a_n z^n$ may diverge, may converge nonabsolutely, or may converge absolutely.

(In fact, you can find a power series that shows two of the above phenomena at different values of z with $|z| = R$; but an example of the remaining phenomenon requires a different power series. Do you see why?)

3.13. ADDITION AND MULTIPLICATION OF SERIES. (pp.72-75)

Relevant exercise in Rudin:

3:R13. *Cauchy products of absolutely convergent series.* (**d**: 2)

Exercises not in Rudin:

3.13:0. *Say whether the following statement is true or false.*

(a) If $\Sigma a_n = A$ and $\Sigma b_n = B$, and these series converge absolutely, then $\Sigma a_n b_n = AB$.

3.13:1. *Radii of convergence of sum and product series.* (**d**: 2. >**3.12:2**)

Suppose Σa_n and Σb_n are series, and let their sum in the sense of Theorem 3.47 and their product in the sense of Definition 3.48 be denoted Σs_n and Σp_n respectively.

(a) Show that if r is a real number such that $\Sigma a_n z^n$ and $\Sigma b_n z^n$ both have radius of convergence $\geq r$, then $\Sigma s_n z^n$ and $\Sigma p_n z^n$ also have radii of convergence $\geq r$.

(Suggestion: Don't use the ''lim sup'' formula for the radius of convergence, but its characterization in terms of where power series converge and where they diverge, together with the result of **3.12:2**(a).)

(b) Deduce from (a) that if $\Sigma a_n z^n$ and $\Sigma b_n z^n$ have different radii of convergence, then the radius of convergence of $\Sigma s_n z^n$ is equal to the smaller of those two radii.

(c) Also deduce from (a) that if the radius of convergence of $\Sigma p_n z^n$ is less than that of $\Sigma a_n z^n$, then the radius of convergence of $\Sigma b_n z^n$ is $\leq$ that of $\Sigma p_n z^n$.

3.14. REARRANGEMENTS. (pp.75-78)

Relevant exercises in Rudin: None

Exercises not in Rudin:

3.14:0. *Say whether each of the following statements is true or false.*

(a) There is a rearrangement of the series $\Sigma (-1)^n n^{-1}$ which converges to -2011.

(b) If a series Σa_n has the property that all of its rearrangements converge, it is absolutely convergent.

(c) If a series Σa_n has the property that some rearrangement converges, then it itself converges.

3.14:1. *Two ways of showing Rudin's rearrangement of* $\Sigma (-1)^{n+1}/n$ *converges.* (**d**: 3)

(a) Show that if (a_n) is a sequence such that $\lim_{n\to\infty} a_n = 0$, then $\Sigma_{n=0}^{\infty} a_n$ converges if and only if $\Sigma_{k=0}^{\infty} (a_{2k} + a_{2k+1})$ converges, and that these infinite sums are then equal. In other words, one can test such a series for convergence, and find its sum if this exists, by doing the same for the series gotten by collecting these pairs of successive terms.

Answers to True/False question **3.11:0.** (a) F. (b) T. **Answer** to True/False question **3.12:0.** (a) F.

(You should give a careful ''ε''-proof, unless you prove it using some previous result in Rudin.)

(b) Obtain the analogous result relating $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{k=0}^{\infty} (a_{3k} + a_{3k+1} + a_{3k+2})$. Rather than repeating the whole proof, just indicate carefully what changes need to be made in the proof of (a).

(c) Use (b) above to show that the series (23) on p.76 of Rudin converges. (First describe that series precisely.)

(d) Obtain a different proof of the convergence of Rudin's series (23) by applying Theorem 3.42 with (a_n) the sequence ''1, 1, −2, 1, 1, −2, 1, 1, −2, ...''. (You should figure out what (b_n) is to be.)

(e) Show that the result of (a) becomes false if the condition $\lim_{n\to\infty} a_n = 0$ is removed.

3.14:2. *Which series have convergent rearrangements?* (**d**: 3)

Find a simple criterion for a series $\Sigma\, a_n$ of real numbers to have the property there exists a rearrangement $\Sigma\, a_{k_n}$ which converges.

(The answer to this question is fairly easy; the answer to the corresponding question for series of complex numbers, equivalently, of points of R^2, is much more difficult to state and prove.)

3.14:3. *Which rearrangements of terms don't affect convergence of any series?* (**d**: 1,3,4,5)

We saw in Theorem 3.54 that rearranging the terms of a non-absolutely convergent series can change its behavior drastically. But not all rearrangements can have such effects. Parts (a)-(c) below show that the effects of certain rearrangements are limited in one way or another. Part (d), which is much harder, gives a general criterion for when this happens.

(a) If (a_n) is a sequence, consider the rearrangement gotten by interchanging successive pairs a_{2m} and a_{2m+1}. Clearly, this can be written (a_{k_n}), where (k_n) is the sequence of integers 2, 1, 4, 3, 6, 5, ... , defined by $k_{2m-1} = 2m$, $k_{2m} = 2m-1$ $(m = 1, 2, \ldots)$.

Show that if $\Sigma\, a_n$ or $\Sigma\, a_{k_n}$ is a convergent series, then so is the other, and $\Sigma\, a_{k_n} = \Sigma\, a_n$.

(b) If (a_n) is a sequence, consider the rearrangement gotten by breaking it into blocks whose lengths are successive powers of 2, i.e., $(a_{2^i}, \ldots, a_{2^{i+1}-1})$, and within each such block, collecting all the even-subscripted terms before the odd-subscripted ones, but otherwise preserving their order. This rearrangement can be written (a_{k_n}), for an appropriate sequence (k_n) of integers, whose first 16 terms are as follows. (I use extra space to make visible the separation into ''blocks''.)

$$1, \quad 2, 3, \quad 4, 6, 5, 7, \quad 8, 10, 12, 14, 9, 11, 13, 15, \quad 16, \ldots .$$

Explicitly, for $0 \le i$ and $0 \le j < 2^i$ we have $k_{2^i+2j} = 2^i + j$, and (if $i > 0$) $k_{2^i+2j+1} = 2^i + 2^{i-1} + j$.

Find a convergent series $\Sigma\, a_n$ such that $\Sigma\, a_{k_n}$ diverges.

(c) On the other hand, show that for (k_n) as in part (b) above, if $\Sigma\, a_{k_n}$ converges then $\Sigma\, a_n$ also converges and $\Sigma\, a_{k_n} = \Sigma\, a_n$.

(It follows that if we write (j_n) for the permutation inverse to (k_n), i.e., for each n let j_n be the unique integer such that $k_{j_n} = n$, then the sequence (j_n) has the opposite properties: if $\Sigma\, a_n$ converges then so does $\Sigma\, a_{j_n}$, and $\Sigma\, a_{j_n} = \Sigma\, a_n$, but there exists a divergent series $\Sigma\, a_n$ such that $\Sigma\, a_{j_n}$ converges.)

(d) Now let (k_n) be an arbitrary sequence of positive integers in which each positive integer occurs once and only once. (This could be described in Rudin's language as a *rearrangement* of the sequence of positive integers; or, in different terminology, as a *permutation* of the positive integers.) For every positive integer N, let us define the *mixing number* $\mathrm{mix}((k_n), N)$ to be the largest integer M for which there exist M integers $n_1, n_2, \ldots\, n_M$ which are *alternately* $\le N$ and $> N$ (i.e., such that if $n_i \le N$ then $n_{i+1} > N$ and vice versa), and such that $k_{n_1} < k_{n_2} < \ldots < k_{n_M}$. In other words, if we color each positive integer m red or blue according to whether it shows up before or after the Nth comma in the sequence

Answer to True/False question **3.13:0.** (a) F. **Answers** to True/False question **3.14:0.** (a) T. (b) T. (c) F.

$k_1, k_2, k_3, \ldots,$ then $\mathrm{mix}((k_n), N)$ denotes the number of same-color *blocks* into which the set of all positive integers is divided. Since only N integers are colored red, this number is at most $2N+1$; in particular, it is finite.

If you think through the example you constructed for part (b), you should find that it implicitly used the fact that for the sequence of integers occurring there, $\mathrm{mix}((k_n), N)$ assumed arbitrarily large values, while the positive result of (c) was based on the fact that $\mathrm{mix}((j_n), N)$ never goes higher than 4.

In fact, prove that the following conditions on a rearrangement (k_n) of the positive integers are equivalent:

(i) $\mathrm{mix}((k_n), N)$ is bounded as a function of N.

(ii) For every convergent series $\Sigma\, a_n$ of real numbers, the series $\Sigma\, a_{k_n}$ also converges.

Show moreover that if these conditions hold, then for every convergent series $\Sigma\, a_n$ of real numbers, $\Sigma\, a_{k_n} = \Sigma\, a_n$.

3.14:4. *Rearrangements and the root test.* (**d**: 3)

Find a series which converges by the root test, and a rearrangement of this series for which the root test gives no information.

However, note a reason why, in a situation of this sort, the rearranged series must still converge.

3.14:5. *The subsequential limit set of a rearranged convergent series.* (**d**: 1)

Let $\Sigma\, a_n$, α and β be as in the hypothesis of Theorem 3.54, and let $\Sigma\, a'_n$ and (s'_n) be as in the conclusion thereof. Assuming the result of **3.2:6**(a) (even if you haven't proven it), deduce that the subsequential limit set of (s'_n) is the interval $[\alpha, \beta]$.

3.14:6. *Rearrangements with alternating signs.* (**d**: 5)

Show that if (a_n) is a non-absolutely convergent series, and α is a real number, then there exists a rearrangement (a_{k_n}) of (a_n) such that for all even n, $a_{k_n} \geq 0$, for all odd n, $a_{k_n} \leq 0$, and $\Sigma\, a_{k_n} = \alpha$.

(One could even get the partial sums to have $\limsup$ any α and $\liminf$ any β such that $-\infty \leq \alpha \leq \beta \leq +\infty$, as in Theorem 3.54. But proving that would just be more work, without a really different idea.)

Chapter 4. Continuity.

4.1. LIMITS OF FUNCTIONS. (pp.83-85)

Relevant exercises in Rudin: None

Exercises not in Rudin:

4.1:0. *Say whether each of the following statements is true or false.*

(a) If $f: X \to Y$ is a function between metric spaces, then for every $p \in X$, $\lim_{x \to p} f(x)$ exists.

(b) Let X and Y be metric spaces, E a subset of X, p a limit point of E in X, q a point of Y, and $f: E \to Y$ a function. If for all $x \in X$, $d(f(x), q) \leq 10\, d(x,p)$, then $\lim_{x \to p} f(x) = q$.

(c) Let $X, \ldots, f$ be as in the first sentence of the preceding part. If for all $x \in X$, $d(f(x), q) \leq d(x,p) + 1/10$, then $\lim_{x \to p} f(x) = q$.

(d) Suppose X is a metric space, E a subset of X, p a point of $\overline{E}$, f a real-valued function on E such that $\lim_{x \to p} f(x)$ exists, and c a real number such that for all $x \in E$ we have $f(x) \leq c$. Then $\lim_{x \to p} f(x) \leq c$.

(e) Suppose X is a metric space, E a subset of X, p a point of $\overline{E}$, f a real-valued function on E such that $\lim_{x \to p} f(x)$ exists, and c a real number such that for all $x \in E$ we have $f(x) < c$. Then $\lim_{x \to p} f(x) < c$.

4.1:1. *Condition for a product of real-valued functions to approach zero as* $x \to p$. (**d**: 1)

Suppose f and g are real-valued functions on a subset E of a metric space X, and p is a limit point of E. Show that if for some $r > 0$, the set of real numbers $f(E \cap N_r(p))$ is bounded, and if $\lim_{x \to p} g(x) = 0$, then $\lim_{x \to p} f(x)g(x) = 0$. (In words, ‘‘If as $x \to p$, one function remains bounded and the other approaches 0, then their product approaches 0.’’)

4.1:2. *Limit points of functions that don’t necessarily approach a limit.* (**d**: 2)

In Chapter 3, we saw that even if a sequence of points of a metric space did not approach a limit, we could still look at its ‘‘subsequential limit points’’. We can associate a similar set of points to a function between metric spaces:

(a) If $f: E \to Y$ is a function from a subset of a metric space X to another metric space, and p is a limit point of E in X, show that the following three subsets of Y are the same:

(i) $\{y \in Y \mid (\forall \varepsilon > 0)\ (\exists x \in E)\ 0 < d(x,p) < \varepsilon \text{ and } d(f(x), y) < \varepsilon\}$.

(ii) The set of all points $\lim_{n \to \infty} f(x_n)$ such that (x_n) is a sequence converging to p in $E - \{p\}$ and $(f(x_n))$ converges in Y.

(iii) The union, over all sequences (x_n) converging to p in $E - \{p\}$, of the set of subsequential limit points of $(f(x_n))$ in Y.

(b) Show that the set described in three equivalent ways in (a) is closed.

4.1:3. *Analog of the Cauchy criterion for functions between metric spaces.* (**d**: 2, 1)

(a) Suppose $f: E \to Y$ is a function from a subset E of a metric space X to a complete metric space Y, and let $p \in X$ be a limit-point of E. Formulate and prove a necessary and sufficient condition for $\lim_{x \to p} f(x)$ to exist, analogous to the condition ‘‘(s_n) is Cauchy’’ for the limit of a sequence in Y to exist.

(Since ‘‘complete metric space’’ is defined in terms of the convergence of Cauchy sequences, your proof will have to relate the behavior of functions to the behavior of sequences. But the actual formulation of your criterion should be as an ‘‘ε-δ’’ condition, not one about sequences.)

(b) Show that if the assumption that Y is complete is deleted from (a), the condition you have obtained is still necessary, but no longer sufficient, for $\lim_{x \to p} f(x)$ to exist.

4.2. CONTINUOUS FUNCTIONS. (pp.85-89)

Relevant exercises in Rudin:

4:R1. *Is this the same as continuity?* (**d**: 1)

4:R2. *Continuous maps and closures of subsets.* (**d**: 1)

4:R3. *The zero-set of a continuous function is closed.* (**d**: 1)

4:R4. *Continuous maps agreeing on a dense subset are equal.* (**d**: 2)

4:R5. *Extending a continuous real-valued map from a closed subset to all of R.* (**d**: 3)

In the last sentence of this exercise, Rudin remarks that the result remains true ‘‘if R^1 is replaced by any metric space’’. He means R^1 as the space containing the closed set E; not R^1 as the codomain space of our functions. The result is false if the codomain space is taken, for instance, to be $\{0,1\}$; you should not find it hard to get examples showing this.

4:R7. *Discontinuous functions on R^2 that are continuous on all lines.* (**d**: 2)

If we call the exercise as Rudin gives it part ‘‘(a)’’, we can add

(b) Find continuous functions $a, b: R \to R$ with $a(0) = b(0) = 0$ such that the restriction of the function f of the exercise to the curve $\{(x, a(x)) \mid x \in R\}$ is discontinuous at $(0,0)$, and the restriction of the function g to the curve $\{(x, b(x)) \mid x \in R\}$ is unbounded as one approaches $(0,0)$.

4:R22. *Continuous functions that glide between two closed sets.* (**d**: 1)

The function ρ referred to in this exercise is defined in **4:R20**, but the main result of **4:R20** (which belongs under the next section) is not needed to do this one. (However, having done that exercise can

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Answers to True/False question **4.1:0.** (a) F. (b) T. (c) F. (d) T. (e) F.

shorten the work of this one.)

An explanation of the parenthetical last sentence of this exercise: The concept of ''normality'' which Rudin refers to is defined in the general context of topological spaces; some topological spaces are normal and some are not. Rudin translates the result of the exercise to say is that all *metric spaces* are normal as topological spaces.

4:R23. *Convex functions on an interval.* (**d**: 2)

4:R24. *A test for convexity, using values at midpoints.* (**d**: 3. >**4:R23**)

Exercises not in Rudin:

4.2:0. *Say whether each of the following statements is true or false.*

(a) The function $f: Q \to R$ defined by $f(x) = 0$ if $x^2 < 2$, $f(x) = 1$ if $x^2 > 2$, is continuous.

(b) If p is an isolated point of a metric space X, then every function f from X to a metric space Y is continuous at p.

(c) If $f: X \to Y$ and $g: Y \to Z$ are functions between metric spaces, and $g \circ f$ is discontinuous at a point $p \in X$, then either f is discontinuous at p, or g is discontinuous at $f(p)$.

(d) Under the assumptions of the preceding part, f must be discontinuous at p *and* g must be discontinuous at $f(p)$.

(e) Suppose a map $f: X \to Y$ of metric spaces is one-to-one and onto, so that there exists an inverse map $f^{-1}: Y \to X$. Then if f is continuous, so is f^{-1}.

(f) If X is any metric space, then the identity map $\mathrm{id}_X : X \to X$, defined by $\mathrm{id}_X(x) = x$ for all $x \in X$, is continuous.

(g) If $f: X \to Y$ is a continuous map of metric spaces and E is a bounded subset of X, then $f(E)$ is a bounded subset of Y.

(h) A mapping $f: X \to Y$ of metric spaces is continuous if and only if for every $x \in X$ and every neighborhood N of $f(x)$ in Y, the subset $f^{-1}(N)$ contains a neighborhood of x in X.

(i) If $f: R^2 \to R$ is continuous, then for each $a \in R$, the functions $g, h : R \to R$ defined by $g(y) = f(a, y)$ and $h(x) = f(x, a)$ are continuous.

4.2:1. *Limits of sequences characterized in terms of continuity.* (**d**: 1)

Let (p_n) be a sequence of points in a metric space X, and p a point of X. Let K be the set $\{0\} \cup \{1/n \mid n = 1, 2, 3, ...\} \subseteq R$, and let $f: K \to X$ be the function defined by $f(0) = p$, $f(1/n) = p_n$. Show that $\lim_{n\to\infty} p_n = p$ in X if and only if $f: K \to X$ is continuous.

4.2:2. *Continuity characterized in terms of limits of sequences.* (**d**: 2)

Let $f: X \to Y$ be a map between metric spaces. Show that the following conditions are equivalent:

(i) f is continuous.

(ii) For every sequence (p_n) which converges in X, one has $\lim_{n\to\infty} f(p_n) = f(\lim_{n\to\infty} p_n)$. (Note that the right-hand side of the above equation is defined by assumption; the equation thus means that the left-hand side is defined and equal to the right-hand side.)

(iii) For every sequence (p_n) which converges in X, the sequence $(f(p_n))$ converges in Y.

(Hint for proving (iii)$\Rightarrow$(ii): Given a sequence as in the hypothesis of (ii), with limit q, apply (iii) to a sequence formed by alternately using the terms of the given sequence, and the element q.)

4.2:3. *Connectedness characterized in terms of continuity.* (**d**: 2, 2, 3. >**4:R22**)

(If you don't do this exercise after reading this section, you might do it after section 4.4.)

Let X be a metric space.

(a) Show that if A and B are subsets of X, then A and B are *separated* (Definition 2.45) if and only if there exists a continuous function $f: A \cup B \to R$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

(b) Deduce that X is connected if and only if every continuous function $X \to \{0,1\}$ is constant.

(c) Show likewise that X is connected if and only if every continuous function $X \to Z$ is constant.

4.2:4. *The archimedean property for powers of a continuous increasing function.* (**d**: 4 or 2, 3)

(a) Suppose $f: R \to R$ is a continuous function such that $f(x) > x$ for all x. For any positive integer n, let f^n denote the n-fold composite $f \circ f \circ \ldots \circ f$. Show that for all $x, y \in R$, there exists a positive integer n such that $f^n(x) > y$.

(Part (a) above has difficulty **d**: 4 if you have not done Exercise **1.4:7**, **d**: 2 if you have. Suggestion on how to use that exercise: For each real number x, consider the statement $P(t)$ about a nonnegative real number t saying that for some positive integer n, $f^n(x) > x + t$.)

(b) Does the conclusion of part (a) remain true if the assumption that f is continuous is removed?

4.2:5. *Fixed points for continuous increasing functions.* (**d**: 2)

Let E be a closed bounded subset of R, and $f: E \to E$ a continuous strictly monotone increasing function, i.e., a continuous function such that for all $x, y \in E$, $x < y \Rightarrow f(x) < f(y)$. Take any $x_0 \in E$, and define $x_1, x_2, x_3, \ldots \in E$ by the rule $x_{n+1} = f(x_n)$.

(a) Show that if $x_1 > x_0$, then $x_{n+1} > x_n$ for all n, and sketch why the corresponding implications hold with ''>'' replaced by ''='' and by ''<''.

(b) Show that $\lim_{n \to \infty} x_n$ exists, and is a *fixed point* of f; i.e., that denoting this limit by x we have $f(x) = x$.

(c) Show that if $x_1 > x_0$, then the point x found in (b) is the *least* fixed point of f which is $> x_0$. State the corresponding characterizations of x in the cases $x_1 = x_0$ and $x_1 < x_0$. (You do not have to write out proofs for these cases.)

(Remarks: The same result can be proved if E is the extended real line. Exercise **4.2:4** is related to that fact, though it is not a special case since the function there is not assumed monotone. Another related exercise is **1.2:4**.)

4.2:6. *The descendants of an amoeba.* (**d**: 2. >**4.2:5**)

Let $t \in [0,1]$. Suppose we have a population of amoebas such that every hour on the hour, each amoeba in the population splits in two, and such that over the course of each hour, each amoeba has probability t of surviving, and $1-t$ of dying, with the survival of different amoebas in the population independent of one another. Suppose we start with a single living amoeba at hour 0 (the result of a division that has just taken place, so that it will divide again at hour 1 if it survives till then), and for $n = 0, 1, 2, \ldots$ let p_n denote the probability that at least one of its descendants will be alive at hour n. (We count it as one of its own descendants; thus, $p_0 = 1$.) Since this number depends on the constant t as well as n, let us write it, more precisely, as $p_n(t)$.

(a) Find a formula for computing $p_{n+1}(t)$ from $p_n(t)$.

(Remark: The hard way to approach this question is to look ahead to hour n and consider the number of descendants alive at that time, and whether any will survive during the next hour. The easy way is to note that the original amoeba has probability t of surviving to hour 1, and that if it does so, it will divide and each of the resulting amoebas will have, independently, probability $p_n(t)$ of having at least one living descendant n hours later. Recall also the principle: If two independent events have probabilities x and y of occurring, then the probability that both will occur is xy. To determine the probability that *one or both* will occur, note that this is the probability that they will *not* both fail to occur. The probability that they will both fail to occur is, by the preceding principle, $(1-x)(1-y)$, so the probability that they will *not* is $1 - (1-x)(1-y)$.)

(b) Does $\lim_{n \to \infty} p_n(t)$ exist? If so, determine its value. (Suggestion: Use **4.2:5**.)

Tangential remarks: The assumption of synchronized reproduction is, of course, an absurd

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Answers to True/False question **4.2:0.** (a) T. (b) T. (c) T. (d) F. (e) F. (f) T. (g) F. (h) T. (i) T.

simplification; and one can also see that conditions can't remain constant when $n \to \infty$, since an expanding population will run out of food and space. Nevertheless, the result you will obtain above is probably a reasonable first approximation to the probability that a simple organism entering a new habitat will succeed in proliferating rather than dying out.

Actually, I didn't come upon this question by thinking about amoebas. Rather, I was considering the algebraic situation where one has a binary operation $*$, not necessarily associative, on a set X, and one wants to study expressions in which $*$ connects an arbitrary number of elements, for instance, $(a*b)$, $((a*(b*c))*(d*e))$, etc.. To study ''statistical'' properties of such expressions, I wanted to assume them generated in a random way. In the expressions in question, the elements a, b, c etc. were to appear in fixed order, so when generating the expressions, one could use a place-holder $\square$ instead of these letters, and work with symbols like $((\square*(\square*\square))*(\square*\square))$. I decided that the mathematically simplest sort of random generation would be to start with a single symbol $\square$, and execute a series of steps, at each of which every $\square$ would either be replaced by $(\square*\square)$, with some probability t, or else be declared ''final'', and undergo no more changes after that. When all squares have become ''final'', one has a randomly generated expression. But can one expect all squares to eventually become ''final''? I worked out the probability that this would happen as a function of t – and found that the resulting computation was an example of something it will be useful to know when we reach Chapter 7 of Rudin. So I reformulated the problem as a more concrete question about amoebas, and have introduced it here in preparation for that later chapter.

Some years later, I learned of still another way of looking at this computation. Consider a system of passageways, beginning at an initial point, where each passageway ends by branching into two further passageways. Suppose each passageway has probability t of being open, and $1-t$ of being blocked. Then the above problem concerns the probability that this system has an infinite unblocked route from the initial point. This is a simple case of the subject of *percolation theory*, which takes its name from the case where the passageways are pores in a solid. Closer to real percolation problems, and more difficult to analyze, are cases where the passageways form infinite ''checkerboard arrays'' in 2 or more dimensions.

4.2:7. *When can one compose limit-statements?* (**d**: 1; 3)

The principle ''If as $x \to p$, $f(x) \to q$, and if as $y \to q$, $g(y) \to r$, then as $x \to p$, $g(f(x)) \to r$'' may appear ''obvious''. But the following example shows that it is false.

(a) Let $f: R \to R$ be defined by $f(x) = x \sin x^{-1}$ if $x \neq 0$, $f(0) = 0$, and let $g: R \to R$ be defined by $g(x) = 0$ if $x \neq 0$, $g(0) = 1$. Show that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but that it is not true that $\lim_{x\to 0} g(f((x))) = 0$.

Why is the apparently obvious principle that we started with false? Because ''$\to q$'' has slightly different meanings in the two statements ''as $x \to p$, $f(x) \to q$'' and ''as $y \to q$, $g(y) \to r$''! In the first it means that $f(x)$ takes on values that become arbitrarily close to q; but in the second, it means that y takes on values getting arbitrarily close to q, *other than the value* q *itself*. In the example above, we see that as $x \to p$ the function $y = f(x)$ has the first of these properties, but not the second.

Obviously, it would be good to know conditions under which we *can* correctly ''compose'' limit statements. Necessary and sufficient conditions are given in

(b) Suppose f is a function from a subset E of a metric space X to a metric space Y, and g is a function from a subset F of Y which contains $f(E)$, to a metric space Z. Let p be a limit point of E in X, q a limit point of F in Y, and r a point of Z, such that $\lim_{x\to p} f(x) = q$ and $\lim_{y\to q} g(x) = r$. Show that the following conditions are then equivalent:

(i) $\lim_{x\to p} g(f((x))) = r$.

(ii) Either $q \in F$ and g is continuous at q, or there exists $\varepsilon > 0$ such that $q \notin f(N_\varepsilon(x) \cap E)$ (i.e., the function f does not take on the value q at points arbitrarily close to p).

4.2:8. *Two different meanings of ''neighborhood'' have similar properties.* (**d**: 3)

What Rudin calls a ''neighborhood'' of a point p of a metric space is nowadays more often called an ''open ball'' about the point. Topologists instead generally define a neighborhood of p to mean any set E such that p is an interior point of E. In this exercise, let us call this concept a ''topologist's neighborhood'', and let the unmodified word have Rudin's meaning. We shall see that the topologist's concept is as convenient as Rudin's for some typical uses of the concept.

(a) Show that if p is a point of a metric space X, then a subset $E \subseteq X$ is a ''topologist's neighborhood'' of p if and only if it contains a neighborhood of p in Rudin's sense.

(b) If $f: X \to Y$ is a map between metric spaces and p a point of X, show that the following conditions are equivalent:

(i) The map f is continuous at p.

(ii) The inverse image under f of every neighborhood of $f(p)$ contains a neighborhood of p.

(iii) The inverse image under f of every ''topologist's neighborhood'' of $f(p)$ is a ''topologist's neighborhood'' of p.

(c) State similarly how the concepts of a limit-point of a set and the limit of a sequence can be formulated in terms of the topologist's concept of neighborhood. (No proofs asked for.)

4.2:9. *At-most-countably-many-to-one continuous maps, and perfect sets.* (**d**: 3. > **2.4:2**)

Let us call a function $f: X \to Y$ ''at most countably many to one'' if for every $y \in Y$, the set $f^{-1}(y) \subseteq X$ is at most countable. Suppose f is a continuous, at most countably many to one map of metric spaces. Show that for every compact perfect subset $E \subseteq X$, $\overline{f(E)}$ is a perfect subset of Y.

(This exercise might be given with one of the next two sections, since the above result nicely parallels the results of those sections on how continuous maps behave on compact sets and connected sets.)

4.2:10. *k-dimensional space-filling curves.* (**d**: 2, 1, 3)

In **7:R14** (p.168), Rudin will show how to construct a continuous map Φ from $[0,1]$ *onto* $[0,1]^2 = \{(x,y) \mid x,y \in [0,1]\}$. This is called a ''space-filling curve'' because a continuous function from $[0,1]$ into the plane can be thought of as a parametrized curve, and this curve fills up all the space inside the square $[0,1]^2$. *Assuming this result,* we shall show here, using only the methods of this section, that there must also exist curves filling higher-dimensional boxes, and even something that can be looked at as an infinite-dimensional space-filling curve.

Given a continuous map Φ from $[0,1]$ onto $[0,1]^2$, let us write $\Phi(t) = (a(t),\, b(t))$ for each $t \in [0,1]$. (Rudin writes $(x(t),\, y(t))$, but I will be using x and y for real numbers.) Thus, a and b are continuous functions $[0,1] \to [0,1]$ such that for every point $(x,y) \in [0,1]^2$ there exists $t \in [0,1]$ such that $(a(t),\, b(t)) = (x,y)$.

(a) Assuming such functions a and b given, deduce that for every point $(x,y,z) \in [0,1]^3$ there exists $t \in [0,1]$ such that $(a(t),\, a(b(t)),\, b(b(t))) = (x,y,z)$. Conclude that the function taking t to $(a(t),\, a(b(t)),\, b(b(t)))$ is a continuous function from $[0,1]$ onto $[0,1]^3$ (a 3-dimensional space-filling curve).

(b) Generalize the argument of part (a) to show that if we write b^i for the i-fold composite function $b \circ b \circ \ldots \circ b$ (with i factors), then for every $n \geq 2$ the function $[0,1] \to [0,1]^n$ taking t to $(a(t),\, a(b(t)),\, \ldots,\, a(b^{n-2}(t)),\, b^{n-1}(t))$ is continuous and onto (an ''n-dimensional space-filling curve'').

(Remark: In Rudin, for f a real-valued function, f^n usually means the function defined by $f^n(x) = f(x)^n$. Hence that notation holds in this exercise packet *unless* the contrary is stated. In this exercise, I have stated the contrary!)

(c) Deduce that for every sequence (x_n) of elements of $[0,1]$, there exists $t \in [0,1]$ such that

$$a(b^{n-1}(t)) \;=\; x_n \quad \text{for all } n \geq 1.$$

(Recall that a *sequence* (x_n) involves infinitely many terms, $x_1,\, x_2,\, \ldots$. We understand b^0 to denote the identity map of $[0,1]$, given by $b^0(t) = t$ for all t. Thus, the $n = 1$ and $n = 2$ cases of the above equation are $a(t) = x_1$ and $a(b(t)) = x_2$.)

Hint: Deduce from part (b) that for every N, the set of t for which the first N of the above equations hold is nonempty. Then use compactness.

Exercise **4.2:14** below will show that the above result can be looked at describing an ‘‘infinite-dimensional space-filling curve’’.

4.2:11. *Plane-filling curves*. (**d**: 2)

As in **4.2:10** above (see first paragraph thereof), we will assume here the result of **7:R14**. Deduce that there also exists a continuous map from R onto R^2. (Suggestion: Let your curve send $[0,1]$ onto $[0,1]^2$ by the map given by Rudin, and send various other intervals $[n, n{+}1]$ onto larger and larger squares.)

Remarks: As in part (b) of the preceding exercise, one can similarly obtain surjective continuous maps $R \to R^k$ for all positive integers k. Since this is straightforward, I don't make it an exercise. On the other hand, part (c) of the preceding exercise uses the compactness of $[0,1]$, but R is not compact. This leads to an obvious question, which is answered in the next exercise.

4.2:12. *No infinite-dimensional analog of preceding exercise*. (**d**: 2)

Show that there does not exist a sequence of continuous functions $f_1, f_2, \ldots : R \to R$ with the property that for every sequence (x_n) in R there exists $t \in R$ such that $f_n(t) = x_n$ for all $n \geq 1$. (Hint: Given a sequence of continuous functions (f_n), show that if you choose x_1 sufficiently large, no $t \in [-1,1]$ can satisfy $f_1(t) = x_1$; that if you choose x_2 sufficiently large, no $t \in [-2,2]$ can satisfy $f_2(t) = x_2$, etc..)

4.2:13. *The results of the preceding exercises for* R *imply the same results for* $[0,1)$. (**d**: 2. >**4.2:11**, **4.2:12**)

We start with a fact not based on the preceding exercises:

(a) Show by example that there exist continuous maps from R onto $[0,1)$ and continuous maps from $[0,1)$ onto R.

(b) Deduce from part (a) and the result of **4.2:11** that there exists a continuous function from $[0,1)$ onto $[0,1)^2$.

(c) Deduce from part (a) and the result of **4.2:12** that there does not exist any infinite sequence of continuous functions $g_1, g_2, \ldots : [0,1) \to [0,1)$ with the property that for every sequence (x_i) in $[0,1)$ there exists $t \in [0,1)$ such that $g_n(t) = x_n$ for all $n \geq 1$.

4.2:14. *A sequence of real-valued continuous functions is equivalent to one function into the space of sequences*. (**d**: 3)

Rudin shows in Theorem 4.10 (p.87) that a family of k maps, $f_1, \ldots, f_k$, from a metric space X into R are all continuous if and only if single map $\mathbf{f} : X \to R^k$ given by $\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$ is continuous. We shall describe here a similar result for infinite sequences of maps (f_i).

For simplicity, we will begin with the case of [0,1]-valued functions.

Let $[0,1]^J$ denote the set of all sequences (x_i) of elements of $[0,1]$. Given $(x_i), (y_i) \in [0,1]^J$, define

$$d\big((x_i), (y_i)\big) \;=\; \Sigma_i \, |x_i - y_i| / 2^i .$$

(a) Show that this function d is a metric on $[0,1]^J$.

(b) Show that if X is any metric space and (f_i) is a sequence of functions $X \to [0,1]$, then each f_i is continuous if and only if the map $f: X \to [0,1]^J$ defined by $f(x) = (f_1(x), f_2(x), \ldots)$ is continuous, relative to the above metric on $[0,1]^J$.

(c) Why can the above formula not be used to define a metric on R^J? Show, however, that by replacing $|x_i - y_i|/2^i$ with $\min(|x_i - y_i|, 2^{-i})$, one can get a metric on R^J, and a result analogous to (b) above.

4.2:15. *The set where two continuous functions agree is closed*. (**d**: 2,1,1,1)

(a) Let X and Y be metric spaces, and let $f: X \to Y$ and $g: X \to Y$ be continuous functions. Show that the set $E = \{x \in X \mid f(x) = g(x)\}$ is closed in X. (This can be done either by verifying directly that E satisfies the definition of a closed set, or by showing that the complement of E is open, or if one has done **4:R3**, by showing that the function taking $x \in X$ to $d(f(x), g(x)) \in R$ is continuous, and noting that E is the zero-set of this function.)

(b) Show that (a) implies the result of **4:R3**.

(c) Deduce the main result of **4:R4** – the final non-parenthetical statement, beginning ''If $g(p) = f(p)$ ...'' – from (a).

(d) Assuming the results of **4:R20** (whether or not you have done that exercise), prove the following converse to (a) above: For every closed subset E of a metric space X, there exist continuous functions f and g from X to another metric space Y such that $\{x \in X \mid f(x) = g(x)\} = E$.

4.2:16. *Functions which approach a limit everywhere.* (**d**: 3)

Suppose $f: X \to Y$ is a function between metric spaces (not assumed continuous) such that for every $p \in X$, $\lim_{x \to p} f(x)$ exists. Define $g: X \to Y$ by $g(p) = \lim_{x \to p} f(x)$ for all $p \in X$. Show that g is continuous.

(There are several ways this result can be strengthened. The above assumption that the limit of f is defined at every $p \in X$ necessitates that X have no isolated points. One can weaken this to say that the limit of f exists at every non-isolated point of X, defining g as above at those points, while making it agree with f at the isolated points; one can then establish the same conclusion as above. One can also merely assume f to be defined on a dense subset E of X; this still allows it to have a limit at every non-isolated point of X, so that one can define g on all of X as above, and again get the same conclusion.)

4.3. CONTINUITY AND COMPACTNESS (and uniform continuity). (pp.89-93)

Relevant exercises in Rudin:

4:R6. *A function on a compact space is continuous if and only if its graph is compact.* (**d**: 2)

4:R8. *A uniformly continuous function on a bounded subset of R^n is bounded.* (**d**: 3)

4:R9. *Uniform continuity in terms of diameters of sets.* (**d**: 1)

In this exercise, for Rudin's phrase ''the requirement in the definition of uniform continuity'' simply read ''the condition that $f: E \to Y$ be uniformly continuous''.

4:R10. *Alternative proof of Theorem 4.19.* (**d**: 2)

4:R11 and **4:R13**. *Extension of real-valued functions from dense subsets.* (**d**: 1,3 and 3)

These two exercises ask you to prove the same result, but by different methods. The statement of this result is given in the later exercise, to which the second sentence of the earlier exercise refers you. The first sentence of the earlier exercise is an easy but instructive result on Cauchy sequences.

4:R12. *A uniformly continuous function of a uniformly continuous function is uniformly continuous.* (**d**: 1)

4:R20. *The distance from a set is a uniformly continuous function.* (**d**: 2)

4:R21. *The distance from a compact set to a disjoint closed set is bounded below.* (**d**: 3. >**4:R20**)

In the last sentence, ''Show that the conclusion may fail'' means, of course, give an example where the conclusion fails.

4:R25. *The ''sum'' of a compact set and a closed set is closed.* (**d**: 3. >**4:R21**)

4:R26. *Properties of a map which factors through a continuous one-to-one map on a compact set.* (**d**: 3)

Exercises not in Rudin:

4.3:0. *Say whether each of the following statements is true or false.*

(a) A function $\mathbf{f}$ from a subset E of a metric space X to R^k is bounded if and only if $\mathbf{f}(E)$ is a bounded subset of R^k.

(b) If f is a continuous function from R to a metric space X, then for every positive integer n, $f([-n, n])$ is a compact subset of X.

(c) Let f be a continuous real-valued function on a metric space X. If there exists a point $p \in X$ such that $f(p) = \sup_{x \in X} f(x)$, then X is compact.

(d) The function $f: (0, 1) \to R$ defined by $f(x) = 1/x$ is uniformly continuous.

(e) The function $f: (1, +\infty) \to R$ defined by $f(x) = 1/x$ is uniformly continuous.

(f) The function $f: (0, +\infty) \to R$ defined by $f(x) = 1/x$ is uniformly continuous.

(g) If $f: R \to R$ is uniformly continuous, then f assumes a maximum value, i.e., there exists a real number a such that $f(a) = \sup_{x \in R} f(x)$.

(h) If X is a noncompact metric space, then there exists a real-valued function f on X which is uniformly continuous but not continuous.

(i) A subset $E \subseteq R$ is compact if and only if every continuous real-valued function on E is bounded.

4.3:1. *On what metric spaces is every function continuous, respectively uniformly continuous?* (**d**: 3)

(a) What metric spaces X have the property that every function from X to any metric space Y is continuous?

(b) What metric spaces X have the property that every function from X to any metric space Y is *uniformly* continuous?

4.3:2. *Continuous periodic functions on R are uniformly continuous.* (**d**: 2)

(a) Let c be a positive real number, and $f: R \to X$ a continuous function from R to a metric space X which has period c, i.e., such that $f(r+c) = f(r)$ for all $r \in R$. Show that f is uniformly continuous.

So, for instance, the function $\sin x$ is uniformly continuous. However

(b) Show that the function $\sin(x^2)$ is not uniformly continuous. (Rudin has not developed the sine function as of this point. However, all you need to know to do (b) is that $\sin x$ is a nonconstant periodic continuous real-valued function on R.)

4.3:3. *A continuous map on R^k takes bounded sets to bounded sets.* (**d**: 2)

Let X be a metric space and $f: R^k \to X$ a continuous function. Show that if E is a bounded subset of R^k, then $f(E)$ is a bounded subset of X.

4.3:4. *A shrinking map on a compact space has a fixed point.* (**d**: 4, 2, 2, 3, 4, 4)

(a) Suppose K is a compact metric space, and $f: K \to K$ is a map with the property that for every pair of distinct points $x, y \in K$ one has $d(f(x), f(y)) < d(x, y)$. Show that there exists a unique point $p \in K$ such that $f(p) = p$.

(The remaining parts, though of interest, can be omitted without detracting from the interest or challenge of the above problem.)

(b) Show by example that if the ‘‘<’’ is weakened to ‘‘≤’’ in part (a), the map need not have any fixed point.

(c) Show by examples that the result of (a) is also false if we put any of the noncompact metric spaces $[0,1)$, $[0, +\infty)$ or R in place of the compact space K.

(d) Deduce from (a) that for K a compact metric space, there cannot exist a continuous function f: $K \to K$ such that for all distinct points $x, y \in K$ one has $d(f(x), f(y)) > d(x, y)$.

(e) Show that in the situation of (a), for every $x \in K$ one has $\lim_{n \to \infty} f_n(x) = p$.

(f) Suppose K is a compact subset of a metric space X, and $g: K \to X$ is a map such that $g(K) \supseteq K$, and such for every pair of distinct points $x, y \in K$ one has $d(g(x), g(y)) > d(x, y)$. Show that there exists a unique point $p \in K$ such that $g(p) = p$.

(For some further related results, see **4.3:8**.)

4.3:5. *Uniform ‘‘either/or continuity’’ of two or more functions.* (**d**: 3)

(a) Let X, Y and Z be metric spaces, with X compact, and let $f: X \to Y$, $g: X \to Z$ be functions. We shall not assume f and g are continuous, but let us assume that for each $p \in X$, at least one of f

and g is continuous at p. Show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $d(p,q) < \delta$ in X, one has either $d(f(p),f(q)) < \varepsilon$ or $d(g(p),g(q)) < \varepsilon$.

(b) Does a similar result hold for an infinite family of maps $f_i : X \to Y_i$ $(i \in I)$? More generally, if we have such a family of maps, and also an arbitrary family of positive real numbers ε_i $(i \in I)$, can we find a single δ such that whenever $d(p,q) < \delta$ in X, there exists some $i \in I$ such that $d(f_i(p),f_i(q)) < \varepsilon_i$?

4.3:6. *A weak condition that implies boundedness.* (**d**:3,1,2)

(a) Suppose E is a subset of a compact metric space X, and f is a function from E to a metric space Y which satisfies the following condition much weaker than uniform continuity: For *some* $\varepsilon > 0$ there exists $\delta > 0$ such that all $p,q \in X$ that satisfy $d_X(p,q) < \delta$ also satisfy $d_Y(f(p),f(q)) < \varepsilon$. Show that f must be bounded. (Note: f is not assumed continuous. Hint: Construct a certain open covering of $\overline{E}$.)

(b) Deduce **4:R8** from (a).

(c) Find a bounded (but, necessarily, non-compact) metric space X and a uniformly continuous function f from X to another metric space Y such that f is not bounded.

4.3:7. *More on functions which approach a limit everywhere.* (**d**:4)

Suppose $f: X \to Y$ is a function between metric spaces (not assumed continuous) such that for every $p \in X$, $\lim_{x \to p} f(x)$ exists. (This condition was considered in **4.2:16**, but the present exercise does not depend on that one.) Let $D \subseteq X$ be the set of points where f is discontinuous. Show that for every compact subset $K \subseteq X$, the set $D \cap K$ is countable.

(Suggestion: For $p \in X$, let $w(p) = \lim_{\varepsilon \to 0} \operatorname{diam}(f(N_\varepsilon(p)))$. Show that for every positive constant c, if $\{p \in X \mid w(p) > c\}$ had a limit point $q \in X$, then $\lim_{x \to q} f(x)$ could not exist, a contradiction. Deduce that for each positive integer n there are only finitely many $p \in K$ with $w(p) > 1/n$.)

4.3:8. *Weakly shrinking maps on compact spaces.* (**d**:5)

(a) Suppose K is a compact metric space, and $f: K \to K$ is a map with the property that for all $x,y \in K$ one has $d(f(x), f(y)) \leq d(x,y)$. (Cf. **4.3:4**.) Show that f is surjective if and only if for all $x,y \in K$ one has $d(f(x), f(y)) = d(x,y)$.

(b) Deduce from (a) that if K is a compact metric space and $g : K \to K$ a continuous map with the property that for all $x,y \in K$ one has $d(f(x), f(y)) \geq d(x,y)$, then for all x,y one has $d(f(x), f(y)) = d(x,y)$.

4.4. CONTINUITY AND CONNECTEDNESS. (p.93)

Relevant exercises in Rudin:

4:R14. *Every continuous map* $[0,1] \to [0,1]$ *has a fixed point.* (**d**:3)

(A very brief hint will turn the difficulty-rating of the above exercise from **d**:3 to **d**:1.)

4:R19. *When the intermediate-value property implies continuity.* (**d**:3)

Exercises not in Rudin:

4.4:0. *Say whether each of the following statements is true or false.*

(a) Every continuous map $R \to Q$ is constant.

(b) Every continuous map $Q \to R$ is constant.

Exercise **4.2:3** (listed under section 4.2) is also relevant to this section.

4.4:1. *There's no continuous one-to-one map from the circle to the line.* (**d**:2)

Let $S = \{(x,y) \in R^2 \mid x^2 + y^2 = 1\} = \{(\sin t, \cos t) \mid t \in [0, 2\pi]\}$. Show that there is no continuous one-to-one function $S \to R$. (You may assume the equivalence of the above two characterizations of S, and so use either in proving the result.)

Answers to True/False question **4.3:0.** (a) T. (b) T. (c) F. (d) F. (e) T. (f) F. (g) F. (h) F. (i) T.

4.4:2. *The closed half-line is not like the whole line.* (**d**: 3)

(a) Show that there is no one-to-one continuous map from $[0, +\infty)$ onto R.

(b) Show that there is no one-to-one continuous map from R onto $[0, +\infty)$.

(c) Deduce from (a) and (b) the corresponding results with $[0, 1)$ in place of $[0, +\infty)$.

(Exercise **4.2:13** showed that in some ways $[0, 1)$ was like the real line. This exercise shows that this similarity only goes so far.)

4.4:3. *A space-filling curve cannot be one-to-one.* (**d**: 4)

In some previous exercises we have discussed properties of space-filling curves. (See in particular first paragraph of **4.2:10**.) We shall show here that such a curve cannot be one-to-one.

(a) Let $n > 1$. Show that for every point p of $[0,1]^n$, the subset obtained by removing p from $[0,1]^n$ is still connected. On the other hand, point to a result in Rudin showing that $[0,1]$ does not have this property.

(b) Show how the existence of a one-to-one continuous map from $[0,1]$ onto $[0,1]^n$, together with the above pair of facts, would lead to a contradiction. (Hint: What do you know about one-to-one continuous maps of one compact metric space onto another?)

4.4:4. *Continuous functions on disconnected sets.* (**d**: 2, 1, 3)

In **2.5:1**(a) we saw that a metric space X is connected if and only if the only subsets of X that are both open and closed are X and $\varnothing$.

(a) Deduce from that result that X is disconnected if and only if there exists a family of nonempty open subsets $E_i \subseteq X$ $(i \in I)$ such that $X = \bigcup_{i\in I} E_i$, the sets E_i are pairwise disjoint (i.e., $i \neq j \Rightarrow E_i \cap E_j = \varnothing$), and I has more than one element. (Suggestion: Think first about the case where I has two elements.)

(b) Show that if X and the E_i satisfy the conditions shown in (a), then a function f from X to a metric space Y is continuous if and only if for each $i \in I$, the restriction f of to E_i is continuous. (The restriction of f to E_i means the function $f|_{E_i}: E_i \to Y$ defined by $f|_{E_i}(x) = f(x)$ for all $x \in E_i$.)

(c) Show, conversely, that if $(E_i)_{i\in I}$ is a family of pairwise disjoint subsets of X whose union is all of X, and such that every function f from X to any other metric space Y such that the restriction of f to each E_i is continuous is itself continuous, then all E_i are open.

4.5. DISCONTINUITIES. (pp.94-95)

Relevant exercises in Rudin:

4:R16. *Discontinuities of integer-part and fractional-part functions.* (**d**: 1)

Where Rudin asks "What discontinuities do the functions $[x]$ and (x) have?" read "Find all points at which these functions are discontinuous, determine the right- and left-hand limits of the functions at those points if they exist, state the values of the functions at those points, and say what kinds of discontinuity the functions have at these points".

4:R17. *A real function on an open interval has at most countably many simple discontinuities.* (**d**: 3)

Here conditions (b) and (c) are sloppily stated; they should read

(b) $a < q < x$, and for all $t \in (q, x)$, $f(t) < p$.

(c) $x < r < b$, and for all $t \in (x, r)$, $f(t) > p$.

4:R18. *A function that is continuous at all irrationals and discontinuous at all rationals.* (**d**: 3)

Answers to True/False question **4.4:0.** (a) T. (b) F.

Exercise not in Rudin:

4.5:0. *Say whether each of the following statements is true or false.*

(a) The function $f(x) = 1/x$ has a discontinuity of the second kind at $x = 0$.

(b) The function f such that $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$ has a discontinuity of the second kind at $x = 0$.

(c) Suppose f, g are real-valued functions on R such that $f(x) = g(x)$ for all $x \neq 0$, while $f(0) \neq g(0)$. Then if f is continuous at 0, g must have a discontinuity of the first kind at 0.

(d) Suppose g is a real-valued functions on R which is continuous for $x \neq 0$ and has a discontinuity of the first kind at $x = 0$. Then there is a real-valued function f on R such that $f(x) = g(x)$ for all $x \neq 0$, and f is everywhere continuous.

4.6. MONOTONIC FUNCTIONS. (pp.95-97)

Relevant exercises in Rudin:

4:R15. *Every continuous open map $R \to R$ is monotonic.* (**d**: 3)

If you don't see how to get started on this one, you might look at $f(x) = x^2$, an example of a map $R \to R$ which is *not* monotonic, and observe, by applying it to the segment $(-1,1)$, that it is also not open. Try to show that every map that fails to be monotonic also fails to be open for the same sort of reason.

Exercises not in Rudin:

4.6:0. *Say whether each of the following statements is true or false.*

(a) Every constant function $R \to R$ is both monotonically increasing and monotonically decreasing.

(b) If $f: R \to R$ is any function, then $\{x \in R \mid f \text{ is discontinuous at } x\}$ is a closed set.

4.6:1. *Increasing functions have a property like uniform continuity.* (**d**: 4)

Suppose $\alpha : [a,b] \to R$ is a monotonically increasing function. Show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $a \leq x < y \leq b$ with $y - x < \delta$, either

$$\alpha(y) - \alpha(x) < \varepsilon,$$

or there exists $z \in [x, y]$ such that

$$\alpha(z+) - \alpha(z-) \geq \varepsilon, \quad \text{and} \quad (\alpha(y) - \alpha(z+)) + (\alpha(z-) - \alpha(x)) < \varepsilon,$$

where if $z = a$ we replace $\alpha(z-)$ by $\alpha(z)$ in the above inequalities, while if $z = b$ we replace $\alpha(z+)$ by $\alpha(z)$.

4.7. INFINITE LIMITS AND LIMITS AT INFINITY. (pp.97-98)

Relevant exercises in Rudin: None

Exercises not in Rudin:

4.7:0. *Say whether each of the following statements is true or false.*

(a) If f is a continuous real-valued function on a bounded segment (a, b), then as $x \to b$, either $f(x)$ approaches a real number, or it approaches $+\infty$, or it approaches $-\infty$.

(b) Suppose f is a function $R \to R$ such that as $x \to +\infty$, $f(x)$ either approaches a real number, or approaches $+\infty$, or approaches $-\infty$. Then as $x \to +\infty$, $1/(f(x)^2 + 1)$ approaches a real number.

(c) If $f: R \to R$ satisfies $\lim_{x \to +\infty} f(x) = a$ for some real number a, then the sequence of real numbers $(f(n))$ converges to a.

(d) If $f: R \to R$ is a continuous function, and if the sequence of real numbers $(f(n))$ converges to the real number a, then $\lim_{x \to +\infty} f(x) = a$.

(e) Suppose $f, g : R \to R$ are functions such that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = -\infty$. Then $\lim_{x \to +\infty} f(x) + g(x)$ does not exist.

(f) Suppose $f, g: R \to R$ are functions such that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = -\infty$. Then $\lim_{x\to+\infty} f(x) + g(x)$ exists.

4.7:1. *A function on* R *that has limits at* $\pm\infty$ *is uniformly continuous.* (**d**: 2)

Let $f: R \to R$ be a continuous function such that $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist. Show that f is uniformly continuous.

(Note that, following Rudin's conventions, the statement that the limits both exist means that they are real numbers. If f approached $+\infty$, as x approached $+\infty$, for instance, Rudin would write "$f(x) \to +\infty$ as $x \to +\infty$", but he would not write "$\lim_{x\to+\infty} f(x) = +\infty$", and would not say that $\lim_{x\to+\infty} f(x)$ exists.)

4.7:2. *A continuous map asymptotic to a uniformly continuous map is uniformly continuous.* (**d**: 2, 3)

For what we want to prove, we first need to strengthen Theorem 4.19:

(a) Let $f: X \to Y$ be a continuous map between metric spaces, and K a compact subset of X. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $p \in K$ and $x \in X$ with $d(p,x) < \delta$, one has $d(f(p), f(x)) < \varepsilon$. (If you wish, you may simply note, in precise detail, changes in the proof of Theorem 4.19 that will yield the above result.)

(b) If $f, g: X \to Y$ are two maps between metric spaces, let us say f and g are "asymptotic" to one another if for every $\varepsilon > 0$, the set $\{x \in X \mid d(f(x), g(x)) \geq \varepsilon\}$ is contained in a compact subset of X. Show that any continuous map which is asymptotic to a uniformly continuous map is uniformly continuous. Then show that the result of **4.7:1** above follows from this result.

4.7:3. *Cauchy criteria for limits of functions.* (**d**: 2)

(a) Suppose f is a complex-valued function on a neighborhood of $+\infty$, $(c, +\infty)$. Show that $\lim_{x\to+\infty} f(x)$ exists if and only if only if for every $\varepsilon > 0$ there exists a real number $M \geq c$ such that for all $x, y \in (M, +\infty)$ one has $|f(x) - f(y)| < \varepsilon$.

State a similar criterion for a complex-valued function defined on a neighborhood of $-\infty$ to have a limit as $x \to -\infty$. (Since the proof is almost identical, I don't ask you to write it out.)

(b) Suppose f is a complex-valued function defined on a subset E of a metric space X, and let p be a limit point of E in X. State and prove an analogous "Cauchy criterion" for $\lim_{x\to p} f(x)$ to exist.

(c) Show by example(s) that one or more of the above criteria fail if "complex-valued function" is replaced by "function into a metric space Y". For which metric spaces Y will those results hold?

Chapter 5. Differentiation.

5.1. THE DERIVATIVE OF A REAL FUNCTION. (pp.103-106)

Relevant exercise in Rudin:

5:R7. *Something that looks like L'Hospital's rule.* (**d**: 1)

We won't see L'Hospital's rule itself until section 5.4. This exercise, despite its appearance, can be done using the definition of derivative, without calling on that result.

5:R13(*a-d*). *Behavior of* $|x|^a \sin(|x|^{-c})$. (**d**: 2)

In this exercise, for $x^a \sin(|x|^{-c})$ read $|x|^a \sin(|x|^{-c})$, since we have not defined what it means to raise a negative number to a non-integer power. In doing this, you may assume standard facts on how the function x^r behaves as $x \to 0$ through positive values; for which real numbers r it is unbounded, for which r it is bounded, and for which r it approaches 0, and likewise you may assume standard formulas for the derivatives of trigonometric functions and functions x^r.

Parts (*e-g*) of the exercise should, strictly speaking, wait till section 5.5, when derivatives of higher order are defined, but since the definition is presumably familiar to everyone, the whole exercise might be

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Answers to True/False question **4.5:0.** (a) F. (b) T. (c) T. (d) F. **Answers** to True/False question **4.6:0.** (a) T. (b) F.

assigned here. It might also be assigned with section 5.3, in view of its relation to the topic of that section.

One could at this point likewise look at the first question asked in **5:R21**, about getting an f which is differentiable; but since the remaining parts of the question refer to higher-order differentiability, I have left that question to section 5.5.

Exercise not in Rudin:

5.1:0. *Say whether each of the following statements is true or false.*

(a) If a bounded function $R \to R$ is continuous, then $f'(t)$ exists for all real numbers t.

(b) If f is a differentiable real-valued function on $[0,2]$ and $\lim_{t\to 1} f(t) = 7$, then $f(1) = 7$.

(c) If f and g are real-valued functions on $[a,b]$, then at any point where f and g are both differentiable, $f+g$ is also differentiable.

(d) If f and g are real-valued functions on $[a,b]$, then at any point where f and $f+g$ are both differentiable, g is also differentiable.

5.2. MEAN VALUE THEOREMS. (pp.107-108)

Relevant exercises in Rudin:

5:R1. *A condition for f to be constant.* (**d**: 2)

5:R2. *Differentiability of inverse functions.* (**d**: 3)

The above is probably not a good exercise to assign for credit, since students owning a good calculus text may be able to find a proof there. On the other hand, it's a result they should know!

5:R3. *Making $x + \varepsilon\, g(x)$ one-to-one.* (**d**: 1)

The instruction to prove that f is one-to-one ‘‘if ε is small enough’’ means, of course, that you are to prove there exists $c > 0$ such that f is one-to-one whenever $0 < \varepsilon < c$.

5:R4. *A condition for a polynomial to have a zero on* $[0,1]$. (**d**: 2)

5:R5. *Functions that change slowly as $x \to +\infty$.* (**d**: 1)

5:R6. *A condition for $f(x)/x$ to be increasing.* (**d**: 3)

To understand this statement, draw a picture, concentrating on conditions (c) and (d), since (a) and (b) are likely to be satisfied by any picture you draw if you don't try to make them fail.

Suggested approach to the exercise: Let $0 < x_1 < x_2$. In your sketch, draw the triangle with vertices (0,0), $(x_1, f(x_1))$, $(x_2, f(x_2))$. To get the desired inequality between the slopes of two sides of this triangle, use the third side.

5:R8. *‘‘Uniform differentiability’’.* (**d**: 2)

Here, apply a result proved in the reading to the fraction in the expression; then see what you can do to insure that what you get is ‘‘close’’ to the term subtracted from that fraction.

The last sentence asks whether the result remains true for vector-valued functions. This must be postponed until you reach section 5.7, so I have repeated the item there.

5:R9. *If the derivative approaches a value at x_0, must it be defined there?* (**d**: 2)

5:R19. *Do difference-quotients $(f(\beta_n) - f(\alpha_n))/(\beta_n - \alpha_n)$ approach f'?* (**d**: 2, 3, 2, 3)

5:R22. *Unique fixed points of differentiable functions $R \to R$.* (**d**: 2, 2, 3, 1)

Part (c) ends by asking you to prove ‘‘that $x = \lim x_n$ where x_1 is an arbitrary real number’’ and a certain relation holds. He means that you should show that for every real number x_1, if one defines x_2 etc. using that relation, then $x = \lim x_n$. For part (d), it is hard to see what sort of answer one could be expected to hand in and have graded; so unless the instructor gives precise instructions on that count, it is probably best to regard this part merely as a suggestion on how to get intuition on the problem.

Answers to True/False question **4.7:0.** (a) F. (b) T. (c) T. (d) F. (e) F. (f) F.

5:R23. *The three fixed points of* $(x^3+1)/3$. (**d**:3)

I have not indicated this exercise as depending on **5:R22**, but if you work on it without having worked that one, you should look that one over and think about parts (c) and (d), so that you know what the author means when he refers to the sequence of numbers x_n determined by a number x_1.

5:R24. *Why some root-finding algorithms converge much faster than others.* (**d**:3)

The student working this exercise should first look over **3:R16**, **3:R17** and **5:R22**, but need not actually have done them.

5:R26. *If* $|f'| \leq A|f|$, *then* f *can't escape from* 0. (**d**:2)

It is easy to see that if a polynomial f has a zero of order exactly n at a point $x = a$, then f' has a zero of order only $n-1$ there. One can look at this as saying that as f ''escapes'' from the value 0 at $x = a$, its derivative must begin by moving away from zero faster than f itself does. This exercise proves a general principle of this sort, in contrapositive form: If f' moves away from zero at a rate no greater than a constant multiple of the value of f, then f in fact never escapes from the value 0.

Though the idea of Rudin's ''Hint'' is right, I find the arrangement messy, and the wording at the end cryptic. I suggest instead letting $c = \sup\{x \mid f(t) = 0 \text{ for all } t \in [a,x]\}$, and applying the method indicated in the *Hint* to show that $f = 0$ on $[a,d]$ for any $d \in [c,b]$ satisfying $d < c+(1/A)$, leading to a contradiction.

The next exercise will show how this one may be applied to the theory of differential equations.

5:R27. *Differential equations with unique and nonunique solutions.* (**d**:3. >**5:R26**)

Note that the sentences after the Hint are a nontrivial part of the exercise.

Exercises not in Rudin:

5.2:0. *Say whether each of the following statements is true or false.*

(a) The function $f(x,y) = x^2 - y^2$ on R^2 has a local maximum at $(0,0)$.

(b) If $f: R \to R$ is differentiable, and x is a point such that $f'(x) = 0$, then f has either a local maximum or a local minimum at x.

(c) If $f: R \to R$ is everywhere differentiable, and satisfies $f(n) = n$ for all integers n, then there are infinitely many real numbers x such that $f'(x) = 1$.

5.2:1. *Which differentiable functions are strictly increasing?* (**d**:2)

(a) Let f be a continuous real-valued function on an interval $[a,b]$ which is differentiable on (a,b). Show that if $f'(x) \geq 0$ for all $x \in (a,b)$, and $f'(x) > 0$ for some $x \in (a,b)$, then $f(b) > f(a)$.

(b) A real-valued function f on a set E of real numbers is said to be *strictly increasing* if for all $x,y \in E$, $x < y \Rightarrow f(x) < f(y)$. Show that a continuous real-valued function f on an interval $[a,b]$ which is differentiable on (a,b) is strictly increasing if and only if it satisfies both (i) $f'(x) \geq 0$ for all $x \in (a,b)$, and (ii) for every pair of points $p < q$ of $[a,b]$ there is an $r \in (p,q)$ such that $f'(r) > 0$.

5.2:2. *A fake counterexample to Theorem 5.9.* (**d**:1)

The conclusion of Theorem 5.9 says, roughly, that if we look at the parametrized curve $(f(t), g(t))$, then for some value of t in (a,b), the ratio of the x- and y-components of its velocity will coincide with the ratio of the x- and y-components of the total displacement, $(f(b) - f(a),\ g(b) - g(a))$; i.e., that there is a point where the direction of the curve is parallel to the line segment from $(f(a), g(a))$ to $(f(b), g(b))$.

However, consider the case $f(t) = t^2$, $g(t) = t^3$ on the interval $[-1,1]$. Sketch the graph, being especially careful near $t = 0$. Is there a point where the curve is parallel to the line segment referred to? For what value of t does the equation of Theorem 5.9 hold? How does the curve look at the corresponding point? (That is why I wrote ''roughly'' in the first sentence of this exercise.)

Answers to True/False question **5.1:0.** (a) F. (b) T. (c) T. (d) T.

5.2:3. *Functions that are differences between two increasing functions*. (**d**: 2, 1, 3)

If g_1 and g_2 are increasing functions on an interval, then their difference, $g_1 - g_2$, need not be either increasing or decreasing. It is natural to ask how wide a class of functions can be written as such a difference. Parts (a) and (b) below show easy classes of examples which can and which cannot be so written; part (c) concerns a more subtle case.

(a) Show that if f is a differentiable function on $[a, b]$ whose derivative is zero at only finitely many points, then f can indeed be written as the difference of two increasing functions.

(b) Show that if f is an unbounded (hence, necessarily, discontinuous) function on $[a, b]$, then it cannot be written as the difference of two increasing functions.

(c) Let f be the function on $[0, 1]$ such that $f(x) = x^2 \sin x^{-2}$ if $x \neq 0$, while $f(0) = 0$. By the same reasoning as in Example 5.6(b) (p.106), $f(x)$ is everywhere differentiable. (This is immediate, so you are not asked to prove it.) However, show that f cannot be written as the difference $g_1 - g_2$ of two increasing functions. (Suggestion: Consider how much f decreases on each interval $[(2\pi n)^{-1/2}, (2\pi(n-1/4))^{-1/2}]$. Sum over n. What can you conclude about g_2?)

5.2:4. *A mean-value theorem with possibly infinite end-points*. (**d**: 3)

Suppose $-\infty \leq a < b \leq +\infty$, and f is a differentiable function on (a, b) such that $\lim_{x \to a} f(x) = \lim_{x \to b} f(x)$. (Note that by Rudin's conventions, writing this entails that the two limits exist, and are real numbers, not $\pm\infty$. On the other hand, the beginning of the above sentence indicates that a and b themselves may be $\pm\infty$.)

Show that there exists $c \in (a, b)$ such that $f'(c) = 0$.

5.2:5. *A condition for* $f(a+)$ *to exist*. (**d**: 2)

Let f be a differentiable function on (a, b). Show that if f' is bounded, then $\lim_{x \to a} f(x)$ exists. (Suggestion: Use **4.1:3**(a) above.)

5.3. Restrictions on discontinuities of derivatives (called by Rudin **THE CONTINUITY OF DERIVATIVES**). (pp.108-109)

Relevant exercises in Rudin: None

Exercises not in Rudin:

5.3:0. *Say whether each of the following statements is true or false*.

(a) If $f: R \to R$ is differentiable, and $f'(x)$ is > 10 for all negative x and < 10 for all positive x, then $f'(0) = 10$.

(b) If $f: (0,1) \cup (2,3) \cup (4,5) \to R$ is differentiable, and f' is negative for all $x \in (0,1)$ and positive for all $x \in (4,5)$, then it must be zero for some $x \in (2,3)$.

5.3:1. *Another restriction on discontinuities of* f'. (**d**: 1)

Suppose f is a real differentiable function on $[a, b]$, let $g = f'$, and let $x \in (a, b]$. Show one cannot have $g(x-) = +\infty$ or $-\infty$.

(I have written $g = f'$ and $g(x-)$ because the symbol $f'(x-)$ might be misunderstood to mean the limit, as t approaches x from the left, of $(f(t) - f(x))/(t - x)$; while what I mean, rather, is the result of applying Definition 4.25 to f'.)

5.4. L'HOSPITAL'S RULE. (pp.109-110)

Relevant exercise in Rudin:

5:R7 could be given here, but as I indicated when I listed it under section 5.1, it can be done without L'Hospital's Rule. For the case of real functions, L'Hospital's Rule gives one possible proof of that exercise. For the case of complex-valued functions L'Hospital's Rule is not available, but the proof from

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Answers to True/False question **5.2:0.** (a) F. (b) F. (c) T.

the definition of derivative still works.

Exercises not in Rudin:

5.4:0. *Say whether each of the following statements is true or false.*

(a) If $f: R \to R$ is a differentiable function satisfying $f'(x) = e^{x^2}$, then $\lim_{x \to +\infty} f(x)/f'(x) = 0$.

(b) If $f: R \to R$ is a differentiable function satisfying $f'(x) = e^{-x^2}$, then $\lim_{x \to +\infty} f(x)/f'(x) = 0$.

5.4:1. *A* ''lim sup'' *version of L'Hospital's rule.* (**d**:3)

For f, g as in the first sentence of Theorem 5.13, and satisfying (14) or (15) of that theorem, adapt the proof of that theorem to get a result relating $\limsup_{x \to a} f(x)/g(x)$ and $\limsup_{x \to a} f'(x)/g'(x)$.

State the result which the corresponding arguments give for lim inf, and show that when (13) holds, these two results together imply (16). (Thus, Theorem 5.13 will follow from these results.)

5.5. DERIVATIVES OF HIGHER ORDER. (p.110)

Relevant exercises in Rudin:

5:R11. *Computing second derivatives with just one limit-operation.* (**d**:3)

Rudin's *Hint* at the end refers to the result you are to prove; so it should come before the sentence asking for an example. Hints for finding the example asked for: If there is such an example, then by subtracting a constant, you can get one where $f(x) = 0$. The easiest way to get the limit to exist is to make the numerator everywhere zero.

5:R12. $|x^3|$ *has some derivatives, but not many.* (**d**:1)

5:R13(*e-g*). *Behavior of* $x^a \sin(|x|^{-c})$ (*continued*). (**d**:1)

See comments under section 5.1 on what you may assume in doing this exercise.

5:R14. *Convexity and* f''. (**d**:2)

The term ''convex function'' used here was defined in **4:R23**, p.101.

5:R21. *Smooth functions with arbitrary zero-sets.* (**d**:4,5)

The rating **d**:4 applies to everything up to the last phrase, about ''derivatives of all orders''. To the student who wishes to attempt that part, I suggest first doing **5.6:1** below.

5:R25(*a,b,f*). *Newton's method: the basics.* (**d**:3)

At the end of the first sentence, where Rudin writes ''$f'(x) \geq \delta > 0$ [...] for all $x \in [a,b]$'', he means ''there exists $\delta > 0$ such that for all $x \in [a,b]$, $f'(x) \geq \delta$''; similarly, the condition on f'' means that there exists an M such that for all x, the stated inequalities hold. In part (*a*), after the first sentence add the instruction ''Show inductively that if x_n is defined and lies in (ξ, b), then the same is true of x_{n+1}''. The last sentence of part (*a*) asks you for a geometric interpretation; this interpretation is not hard to find, but if you don't see it, it is possible for you to do the remaining parts without it.

I discuss parts (*c*) through (*e*) in the next section.

Exercises not in Rudin:

5.5:1. *Derivatives and higher derivatives of bell-shaped curves.* (**d**:3,2,3,4-5. >**5.2:4**)

(a) Let $f: R \to R$ be infinitely differentiable (meaning that $f^{(n)}$ exists for all $n \geq 0$), and suppose that for all $n \geq 0$, $\lim_{x \to +\infty} f^{(n)}(x) = \lim_{x \to -\infty} f^{(n)}(x) = 0$. (In the case $n = 0$, we understand $f^{(0)}$ to mean f.) Show that for each $n \geq 0$ there exist at least n distinct real numbers x such that $f^{(n)}(x) = 0$.

(b) Show that the functions $1/(x^2+1)$, $1/(x^4+1)$ and e^{-x^2} all satisfy the assumptions of (a). (You may assume familiar properties of the exponential function e^x.) Thus by the conclusion of (a), the nth derivative of each of these functions is zero at at least n points of the line.

(c) Show that of the functions named in (b), the first and third have the property that for every n their

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Answers to True/False question **5.3:0.** (a) T. (b) F.

nth derivative is zero at *exactly* n points, but the second does not. (To get a feel for the problem, you might begin by sketching for yourself the functions and their first two or three derivatives. You will be able to prove that exact equality does not hold for one of the functions by finding $> n$ zeroes of $f^{(n)}(x)$ for a particular value of n. In proving that it does hold for the other two, you may use the fact that a polynomial equation of degree d, $a_d x^d + \ldots + a_1 x + a_0 = 0$, has at most d solutions.)

(d) If one looks for other functions f which satisfy the conditions of (a) and whose nth derivative is zero at only n points for every n, one easily sees that this is true of the trivial variants of the above two examples, $A/((Bx+C)^2+1)$ and $A e^{-(Bx+C)^2}$, for all real numbers A, B, C with A and B nonzero. In each case, this is a 3-parameter family of functions. Can you find any functions with these properties that does not belong to one of these families? Any families of such functions with more than three parameters? If so, how large can you make the number of parameters? (By an n-parameter family let us understand a set of functions $f_{A_1, A_2, \ldots, A_n}(x)$ depending on real numbers $A_1, A_2, \ldots, A_n$ such that if $(A_1, A_2, \ldots, A_n) \neq (A_1', A_2', \ldots, A_n')$, then the corresponding functions are distinct.)

5.5:2. *The relation between boundedness of f' and uniform continuity of f.* (**d**: 1,2,3,4)

Let $f: R \to R$ be a differentiable function.

(a) Show that if f' is bounded, then f is uniformly continuous.

(b) Show by example that the converse is not true. (Suggestion: Find a function that is uniformly continuous by **4.7:1**, but whose derivative is unbounded because the function wiggles rapidly for large values of x.)

(c) Show that if an example such as you are asked for in (b) is twice differentiable, then its second derivative must also be unbounded. Equivalently (in view of (a)), show that a twice differentiable function $R \to R$ whose second derivative is bounded is uniformly continuous if and only if its first derivative is bounded.

(d) Can one strengthen (c) to say for every integer $n > 1$ that if $f: R \to R$ is an n times differentiable function which is uniformly continuous, but such that f' is unbounded, then $f'', f^{(3)}, \ldots, f^{(n)}$ must all be unbounded?

5.5:3. *Lagrange interpolation.* (**d**: 3)

(a) Let f be a function on $[a,b]$ which is n times differentiable, i.e., such that $f^{(n)}$ exists, and suppose there are at least $n+1$ points x on $[a,b]$ such that $f(x) = 0$. Show that there is at least one point f on $[a,b]$ such that $f^{(n)}(x) = 0$.

(Suggestion if you have trouble getting started: Draw a sketch of such an f for $n = 3$, and see how many points x in your picture satisfy $f'(x) = 0$. Can you prove there must be that many, or come up with a picture in which there are fewer? Once you can prove something about the number of zeroes of f', see whether you can get from this a conclusion about the number of zeroes of f'', and so on.)

(b) Let f be any function on $[a,b]$ and let $x_0, \ldots, x_n$ be distinct points of $[a,b]$. Show that there exists a polynomial $p(x)$ of degree $\leq n$ such that $p(x_i) = f(x_i)$ for $i = 0, \ldots, n$.

(Hint: If you have found a polynomial that agrees with f at $x_0, \ldots, x_{i-1}$, show that by adding a scalar multiple of $(x-x_0)\ldots(x-x_{i-1})$ you can get a polynomial that agrees with f at $x_0, \ldots, x_i$.)

(c) Let f and p be as in part (b). Because p is a polynomial of degree $\leq n$, its nth derivative $p^{(n)}$ is a constant c. Show with the help of (a) above that if f is n times differentiable, then for some $x \in [a,b]$, $f^{(n)}(x) = c$.

(d) Let f be as in part (b), and let q be the polynomial of degree $\leq n-1$ which agrees with f at the n points $x_0, \ldots, x_{n-1}$. Then the number $E = f(x_n) - q(x_n)$ represents the error resulting when we use this polynomial to approximate f at the n+1st point x_n; also, this number E determines what multiple of $(x-x_0)\ldots(x-x_{n-1})$ must be added to q as in the hint to part (b) to make equality at x_n hold. Write

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Answers to True/False question **5.4:0.** (a) T. (b) F.

down the formula that describes the polynomial of degree $\leq n$ agreeing with f at all $n+1$ points in terms of q and E, take its nth derivative, and apply the result of (c). Conclude that if we know a bound on $f^{(n)}$ on $[a,b]$, then we can bound the error arising when we use q to approximate f at x_n.

(Remark: The polynomial q is called a ‘‘Lagrange interpolation polynomial’’ for f, and the result of (d) is the ‘‘remainder formula for Lagrange interpolation’’.)

5.6. TAYLOR’S THEOREM. (pp.110-111)

Relevant exercises in Rudin:

5:R15. *Bounds relating f, f' and f''.* (**d**: 3)

To motivate this result, consider any twice-differentiable function f on an infinite interval $(a,+\infty)$. If f'' is everywhere zero, then the graph of f is a straight line, so that the only way f can be bounded is if its derivative is everywhere zero. Similarly, if f'' is everywhere very small, then the curve bends very slowly, so if f ever achieves even a moderate nonzero slope, $|f|$ must become large before it can return to a small value. Thus, if the values of $|f|$ and of $|f''|$ are both everywhere small, then that of $|f'|$ must also be. Thus, it is natural to look for an explicit bound for $|f'|$ in terms of $|f|$ and $|f''|$. That is what you obtain here.

Incidentally, here and in the next exercise, when Rudin writes (a,∞), he should, strictly, write $(a,+\infty)$.

5:R16. *Bounded second derivative prevents wriggling near $+\infty$.* (**d**: 2. > **5:R15**)

5:R17. *Specified values at three points lead to a lower bound on the third derivative.* (**d**: 2)

The assumption $f(0) = 0$ is not really needed.

The words ‘‘Note that equality holds for $\frac{1}{2}(x^3+x^2)$’’ simply point out that that function is an example showing that the conclusion ‘‘≥ 3’’ cannot be strengthened to ‘‘$\geq c$’’ for any $c>3$. It does not require you to do anything, unless your instructor tells you to verify that computation.

5:R18. *Taylor’s Theorem with a different error estimate.* (**d**: 3)

I haven’t thought through what is involved in this proof, so my difficulty-estimate is even more of a guess than usual.

5:R25(c,d,e)**.** *Newton’s method: convergence estimates.* (**d**: 3)

Parts (a), (b) and (f) are discussed in the preceding section. In part (c), think carefully about which of x_n, x_{n+1}, ξ, t_n should correspond to which of α, β, x in the statement of Taylor’s Theorem; only one set of choices will give the result you have to prove. In the display in part (d), note that the exponent is 2^n (not $2n$). The last sentence of that part, ‘‘(Compare ...)’’, is for your interest only, not something to write up and hand in. To part (f) add at the end of the first sentence the words ‘‘starting with $x_1 = 1$’’, and add after the last sentence the additional question ‘‘Why does this not contradict what you proved in parts (a) and (b)?’’

Exercises not in Rudin:

5.6:0. *Say whether each of the following statements is true or false.*

(a) If $f: R \to R$ is a function such that $f^{(n)}$ exists for all positive integers n, then $f(x) = \Sigma_{n=0}^{\infty}(f^{(n)}(0)\,x^n)/n!$ for all real numbers x such that this series converges.

(b) If $f: R \to R$ is a function such that $f^{(n)}(x)$ exists and is $\leq 1{,}000$ for all positive integers n and all $x \in R$, then $f(x) = \Sigma_{n=0}^{\infty}(f^{(n)}(0)\,x^n)/n!$ for all $x \in R$.

5.6:1. *A function whose Taylor polynomials converge to the wrong result.* (**d**: 2, 1, 1, 2, 1)

Here is a well-known example concerning Taylor series which Rudin doesn’t give till Chapter 8 (where it is Exercise **8:R1**), because it uses properties of the exponential function, which he develops there. Since we don’t reach that chapter in this course, let us assume for this exercise a few basic properties of that function: That e^x is everywhere nonzero, that $e^{-x} = 1/e^x$, that e^x is differentiable, with derivative equal to itself, and that $e^x \to +\infty$ as $x \to +\infty$. We begin by deducing from these a few more facts.

(a) Show that for every polynomial $p(x)$, as $x \to +\infty$ we have $p(x)\,e^{-x} \to 0$. (Hint: Use L'Hospital's rule and induction on the degree of p.)

(b) Sketch an argument showing from this that for every polynomial $p(x)$, $\lim_{x\to 0} p(x^{-1})\,e^{-x^{-2}} = 0$.

(I say ''sketch'' because, although Rudin proves in Theorem 4.7 that for continuous functions f and g with appropriate domains and codomains, the composite function gf is also continuous, he does not develop general results on *limits* of composite functions; in particular limits at infinity; and although it is not too difficult to prove such results, it would be time-consuming to have you do so here. So simply sketch how these functions behave, assuming that in this example, limits of composites behaves as one would expect. Note that ''$x \to 0$'' involves values both above and below 0.)

We can now give the example. Let $f\colon R \to R$ be defined by $f(x) = e^{-x^{-2}}$ if $x \neq 0$, and $f(0) = 0$.

(c) Show that on $R - \{0\}$, f has derivatives $f^{(n)}$ for all nonnegative integers n, and that for each n, $f^{(n)}(x)$ has the form $p_n(x^{-1})\,e^{-x^{-2}}$ for some polynomial $p_n(x)$.

(d) Show that for every nonnegative integer n, $f^{(n)}(0)$ is also defined, and equals 0. Now compute the Taylor polynomials shown in display (23) on p.110 for $\alpha = 0$. Show that these converge as $n \to \infty$, but that the limit is not the original function $f(t)$.

(e) As a variant of the above, suppose we define a function g by $g(x) = e^{-x^{-2}}$ if $x > 0$, and $g(0) = 0$ if $x \leq 0$. Show that g is also infinitely differentiable, and is given by its Taylor series for all negative x, but not for any positive x.

5.6:2. *Lagrange interpolation with multiplicities.* (**d**: 4. > **5.5:3**)

Suppose f is a function on $[a,b]$, and for some $x \in [a,b]$ and $n \geq 0$, f is $m-1$ times differentiable at x, and $f(x) = f'(x) = \ldots = f^{(m-1)}(x) = 0$. Then one says that f ''has a zero of multiplicity at least m'' at x. (For instance, this is the behavior of a polynomial $p(t)$ that is divisible by $(x-t)^m$.) If there are points $x_1, \ldots, x_r$ in $[a,b]$ and positive integers $m_1, \ldots, m_r$ such that f has a zero of multiplicity at least m_1 at x_1, a zero of multiplicity at least m_2 at x_2, etc., then writing $n = m_1 + \ldots + m_r$, we say that ''f has at least n zeroes on $[a,b]$, *counting multiplicities*''.

Prove the analog **5.5:3**(a) with the condition that f be zero at at least $n+1$ distinct points replaced by the condition that it have at least $n+1$ zeroes counting multiplicities, and use this to get results similarly generalizing parts (b), (c) and (d) of that exercise. Show that Taylor's Theorem (Theorem 5.15, p.110) is a case of the analog of part (d) of that exercise.

5.7. DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS. (pp.111-113)

Relevant exercises in Rudin:

5:R8 (last sentence)**.** *''Uniform differentiability'' for vector-valued functions.* (**d**: 3)

The first part of this exercise was discussed under section 5.2.

5:R10. *A case of L'Hospital's rule valid for complex-valued functions.* (**d**: 3)

5:R20. *Taylor's Theorem for vector-valued functions.* (**d**: 2)

The formula Rudin asks you to get should look like that of Taylor's Theorem, but instead of giving a precise expression for the remainder in terms of the value of the $\mathbf{f}^{(n)}$ at some point, it should give an upper bound in terms of an upper bound on $\mathbf{f}^{(n)}$. You can get this by obtaining such a result for real-valued functions from Taylor's Theorem, and applying it to the components of a vector-valued function.

5:R28. *A uniqueness theorem for systems of differential equations.* (**d**: 3. > **5:R26**, **5:R27**)

5:R29. *A uniqueness theorem for systems of linear differential equations.* (**d**: 3. > **5:R28**)

Answers to True/False question **5.6:0.** (a) F. (b) T.

Exercises not in Rudin:

5.7:0. *Say whether each of the following statements is true or false.*

(a) If $\mathbf{f}, \mathbf{g} : R \to R^k$ are differentiable functions such that for all $x \in R$, $\mathbf{f}(x)\cdot\mathbf{g}(x) = 1$, then for all $x \in R$, $\mathbf{f}'(x)\cdot\mathbf{g}(x) + \mathbf{f}(x)\cdot\mathbf{g}'(x) = 0$.

(b) If $\mathbf{f} : R \to R^2$ is a differentiable function such that $\mathbf{f}(0) = (0,1)$ and $\mathbf{f}(1) = (1,0)$, then there exists c with $0 < c < 1$ such that $\mathbf{f}'(c) = (1,-1)$.

5.7:1. *Derivative of a product of a scalar- and a vector-valued function.* ($\mathbf{d}$: 1)

Rudin mentions on p.112, four lines below display (31), that Theorem 5.3(b) is true for vector-valued functions if ''fg'' is replaced by the inner product ''$\mathbf{f}\cdot\mathbf{g}$''. Prove a version of Theorem 5.3(b) in which fg is replaced by the product $f\mathbf{g}$, where f is a scalar function and $\mathbf{g}$ a vector-valued function. (Here $f\mathbf{g}$ is defined by $(f\mathbf{g})(x) = f(x)\,\mathbf{g}(x)$.)

5.7:2. *Another version of L'Hospital's Rule for complex-valued functions.* ($\mathbf{d}$: 2, 4)

(a) Show that if in Theorem 5.13, we replace $f(x)$ by an R^k-valued function $\mathbf{f}(x)$, and A by a vector $\mathbf{a} \in R^k$, but keep $g(x)$ a real-valued function, then the statement remains true.

(Here a fraction such as $\mathbf{f}(x)/g(x)$ is understood to mean $g(x)^{-1}\mathbf{f}(x)$. Note that the variable x remains real-valued, and a, b continue to be extended reals. We could generalize this exercise by taking $\mathbf{a}$ to be a ''possibly infinite vector'' in the sense of either of **3.4:5** or **3.4:6** above – in fact, this is what led me to put together those exercises! – but I finally decided not to bring that added complication into this one.)

(b) Deduce that Theorem 5.13 becomes true for complex-valued functions (i.e., with f and g complex-valued functions of a real variable x, and A a complex number) if we add the hypothesis that either $\lim_{x\to a} \mathrm{Im}(g'(x))/\mathrm{Re}(g'(x))$ or $\lim_{x\to a} \mathrm{Re}(g'(x))/\mathrm{Im}(g'(x))$ exists. (Clearly this hypothesis does not hold in Rudin's Example 5.18.)

5.7:3. *Disconnected derivative-loci.* ($\mathbf{d}$: 4, 1)

Theorem 5.12 is equivalent to the statement that if f is a differentiable function on an interval, then the set of values of f' is connected.

However –

(a) Show by example that there exists a differentiable function $\mathbf{f}$ from an interval $[a,b]$ to R^2, such that $\{\mathbf{f}'(x) \mid x \in [a,b]\}$ is not connected.

On the other hand –

(b) Show that the Corollary to Theorem 5.12 is true for vector-valued functions; i.e., that if $\mathbf{f}$: $[a,b] \to R^k$ is differentiable, its derivative $\mathbf{f}'$ has no simple discontinuities. (Suggestion: Apply that Corollary to the components of $\mathbf{f}$.)

Chapter 6. The Riemann-Stieltjes integral.

I've made a ''section'' out of the first two pages of this chapter, which discusses the Riemann integral, because the definition of the Riemann-Stieltjes integral involves many concepts difficult for the students, and those two pages ''ease one into'' the subject. I break in two the remainder of Rudin's first section of this chapter, and likewise Rudin's second section, because each is long and contains a number of diverse concepts. After that, the sections below coincide with Rudin's.

6.1. The Riemann integral (beginning of Rudin's section DEFINITION AND EXISTENCE OF THE INTEGRAL). (pp.120-121)

Relevant exercises in Rudin:

6:R2. *The only continuous positive function with integral* 0 *is the zero function.* ($\mathbf{d}$: 2)

Rudin ends by contrasting this with **6:R1**; but that exercise is stated for the Riemann-Stieltjes integral, and so can't be given after just covering these two pages. Hence I give as **6.1:1** below a version of that exercise which refers only to the Riemann integral, and can be assigned in conjunction with this exercise.

6:R4. *The function that is* 1 *at rationals and* 0 *at irrationals is not Riemann integrable.* (**d**: 1)

Exercise not in Rudin:

6.1:1. *A function that is zero except at one point is integrable, with integral zero.* (**d**: 1)

Let f be a function on an interval $[a,b]$ which is zero everywhere except at one point $c\in(a,b)$. Prove that $f\in\mathcal{R}$, and that $\int_a^b f(x)\,dx = 0$.

6.2. The Riemann-Stieltjes integral (middle of Rudin's section **DEFINITION AND EXISTENCE OF THE INTEGRAL).** (pp.122-125)

Relevant exercises in Rudin:

6:R1. *Riemann-Stieltjes integrability of a function zero except at one point.* (**d**: 1)

6:R3. *Integration with respect to three step functions: one right continuous, one left continuous, and one that splits the difference.* (**d**: 2)

I would add to this exercise two more parts:

(e) Given an interval $[a,b]$ and a point $c\in[a,b]$, let $\beta_{j,a,b,c}$ $(j=1,2)$ be the functions on $[a,b]$ which equal 0 on $[a,c)$, 1 on $(c,b]$, and are 0 or 1 at c depending on whether j is 1 or 2. (So for $j=1,2$, Rudin's β_j is $\beta_{j,-1,1,0}$ in this notation.) State the results analogous to Rudin's (a) and (b) for integrals $\int_a^b f(x)\,d\beta_{j,a,b,c}(x)$. Do not hand in proofs. But think things through carefully; you will be graded on the correctness of your answers.

The cases $c = a$ and $c = b$ are slightly different from the general case, so I recommend that you first state the results for $c\in(a,b)$, then say how they must be modified for those cases.

(f) Assuming the results of part (e), deduce that a function f on an interval $[a,b]$ is integrable with respect to *every* increasing function α if and only if it is continuous.

Exercises not in Rudin:

6.2:0. *Say whether each of the following statements is true or false.*

(a) In the equation $U(P,f,\alpha) = \Sigma_{i=1}^{n} M_i\Delta\alpha_i$ (used in Rudin's definition of the Riemann-Stieltjes integral), M_i denotes $f(x_i)$.

(b) In the same equation, n denotes the number of intervals $[x_{i-1},x_i]$ into which the partition P divides $[a,b]$.

(c) If a partition P^* contains more points than a partition P, then P^* is a refinement of P.

(d) If P^* is a refinement of the partition P of the interval $[a,b]$, then for every bounded function f and increasing function α on $[a,b]$, $U(P^*,f,\alpha) - L(P^*,f,\alpha) \leq U(P,f,\alpha) - L(P,f,\alpha)$.

(e) If f is a bounded function and α an increasing function on $[a,b]$, and if there exist partitions P_1, P_2, ... of $[a,b]$ such that for each i, $U(P_{i+1},f,\alpha) - L(P_{i+1},f,\alpha) \leq \frac{1}{2}(U(P_i,f,\alpha) - L(P_i,f,\alpha))$, then $f\in\mathcal{R}(\alpha)$.

(f) If f is a bounded function and α an increasing function on $[a,b]$, and if $P_1 \subseteq P_2 \subseteq \ldots$ are a sequence of partitions, each a refinement of the one before, then $\inf_{n=1,2,\ldots}(U(P_n,f,\alpha)) = \int f\,d\alpha$.

6.2:1. *Riemann-Stieltjes integrability is symmetric.* (**d**: 1)

If α and β are increasing functions on $[a,b]$, show that the following conditions are equivalent: (i) $\alpha\in\mathcal{R}(\beta)$. (ii) $\beta\in\mathcal{R}(\alpha)$. (iii) For every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $\Sigma\,\Delta\alpha_i\,\Delta\beta_i < \varepsilon$, where $\Delta\alpha_i$ and $\Delta\beta_i$ are defined with respect to the partition P as in Definition 6.2 (p.122).

Answers to True/False question **5.7:0.** (a) T. (b) F.

6.2:2. *The Riemann integral as a limit over partitions with mesh approaching* 0. (**d**: 3)

If $P = \{x_0, \ldots, x_n\}$ is a partition of $[a,b]$, let $\|P\| = \sup \Delta x_i$; i.e., the maximum gap between successive points of the partition, often called the ''mesh'' of the partition. In some texts one sees the Riemann integral defined as a limit over partitions whose mesh approaches 0. This exercise will show that such a definition is equivalent to the one given in Rudin. We first need an observation.

(a) Suppose f is a function on $[a,b]$ and M a constant such that $|f(x)| \leq M$ for all x. Show that if P is a partition of $[a,b]$, and P^* a refinement obtained by adding a single point to P, then $U(P,f) - U(P^*,f) \leq 2M\|P\|$, and similarly that $L(P^*,f) - L(P,f) \leq 2M\|P\|$.

(b) Show that the following conditions on a bounded function f on $[a,b]$ are equivalent:

(i) $f \in \mathscr{R}$.

(ii) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition P of $[a,b]$ of mesh $< \delta$, one has $U(P,f) - L(P,f) < \varepsilon$.

Suggestion for (i)⇒(ii): Choose by (i) a partition P_0 for which the difference between the upper and lower sums is $< \varepsilon/2$; then show using part (a) that for any partition P of sufficiently small mesh, $U(P \cup P_0, f) - L(P \cup P_0, f)$ differs from $U(P,f) - L(P,f)$ by $< \varepsilon/2$.

6.3. Conditions for integrability (end of Rudin's section **DEFINITION AND EXISTENCE OF THE INTEGRAL**). (pp.125-127)

Relevant exercises in Rudin:

6:R6. *Discontinuities limited to the Cantor set can't interfere with Riemann integrability.* (**d**: 3)

6:R7. *Improper integrals of the first kind.* (**d**: 2)

6:R8. *Improper integrals of the first kind, and the integral test for convergence of series.* (**d**: 3. > **4.7:3**(a))

The result of this exercise is important, but the exercise is probably not good to give as homework since most students should be able to find proofs, of varying quality, in their lower division calculus texts. One way to get around this would be to hand out a sketchy proof taken from such a text, and ask students to justify specified steps using results from Rudin.

Exercise **4.7:3**(a) supplies a tool that Rudin neglected to develop for showing existence of limits at infinity, which is needed for this exercise.

Exercises not in Rudin:

6.3:0. *Say whether the following statement is true or false.*

(a) If $|f(x)| \leq |g(x)|$ for all $x \in [a,b]$, and $g \in \mathscr{R}(\alpha)$, then $f \in \mathscr{R}(\alpha)$.

6.3:1. *The obvious formula for* $\int 1\, d\alpha$. (**d**: 1)

Show (from the definition of the integral) that for any increasing function α on an interval $[a,b]$, one has $\int_a^b 1\, d\alpha = \alpha(b) - \alpha(a)$. (This should probably have been made part of Theorem 6.12.)

6.3:2. *Functions whose values are ''mostly'' zero have integral zero.* (**d**: 2)

(a) Let $f: [0,1] \to R$ be the function of **4:R18** (p.100) which takes the value 0 at all irrationals, and the value $1/n$ at a rational number whose expression in lowest terms is m/n. Show that for every continuous increasing function α on $[0,1]$ one has $f \in \mathscr{R}(\alpha)$, and $\int_0^1 f\, d\alpha = 0$. (You may do this by doing part (b) below, if you choose.)

(b) Show that the same conclusion is true of any function f on $[0,1]$ with the property that for every $\varepsilon > 0$, the set $\{x \in [0,1] \mid |f(x)| \geq \varepsilon\}$ is finite. Show that the function f of part (a) has that property.

6.3:3. *An* α *which does all its increasing on the Cantor set.* (**d**: 4,3,2,1,3,1)

Let P denote the Cantor set, and let us define a function α_0 on the complement of P in $[0,1]$ as follows. For all x in the segment $(1/3, 2/3)$, i.e., the segment one deletes at the first step in constructing

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Answers to True/False question **6.2:0.** (a) F. (b) T. (c) F. (d) T. (e) T. (f) F.

the Cantor set, let $\alpha_0(x) = 1/2$. At all points of the segments $(1/9, 2/9)$ and $(7/9, 8/9)$, the two segments one deletes at the second stage in the construction of the Cantor set, let $\alpha_0(x)$ have the values $1/4$ and $3/4$ respectively. Similarly, on the 2^{n-1} segments deleted at the nth stage of the construction of the Cantor set, let α_0 have the constant values $1/2^n$, $3/2^n$, ..., $(2^n-1)/2^n$ (each one half-way between the values previously assigned on the two surrounding segments; or in the case of the first and last of these segments, half-way between the value at the adjacent previously assigned segment and the value 0, respectively 1). Now

(a) Show that α_0 can be extended to a function α on all of $[0,1]$ so as to give a continuous increasing function. (Suggestion: Show that the set on which we have defined α_0 is dense in $[0,1]$, and that $\alpha_0(x-)$ and $\alpha_0(x+)$ are defined and equal for all $x\in[0,1]$, and deduce the result using these facts.)

(The function α can also be constructed as follows: Given $x\in[0,1]$, write it in base 3 notation. If it has any digit ''1'', change the first such digit to a ''2'', and *all* digits after it to ''000...''. In the resulting string of 0's and 2's, change all 2's to 1's, and regard the result as an expression for $\alpha(x)$ in base 2. But if you use this description, you must prove it equivalent to the one stated above.)

(b) For α as in part (a), show that if f is a continuous real-valued function on $[0,1]$ which is zero on all points of the Cantor set, then $\int f\,d\alpha = 0$.

(c) Give an example of a function f as in (b) which is not the zero function.

(d) Deduce from (b) that if f and g are two continuous real-valued functions on $[0,1]$ which agree on all points of the Cantor set, then $\int f\,d\alpha = \int g\,d\alpha$.

(e) Let $f: [0,1] \to R$ be the function such that $f(x) = 0$ for all x in the Cantor set, and $f(x) = 1$ for all other x. Show that $f\in\mathcal{R}(\alpha)$.

(f) Why does the result of (e) not contradict the result of (b) above? Why does not it contradict the result of **6:R6** (p.138)?

6.3:4. *An integrable function of an integrable function need not be integrable.* (**d**: 2)

Show by example that, in contrast to Theorem 6.11, if f and φ are Riemann-integrable functions, their composite $\varphi\circ f$ need not be. Suggestion: Let f be as in **4:R18**, and choose φ to be discontinuous at the real number which f most often approaches. (Examples are even known where f is continuous; but they are more difficult to describe.)

6.3:5. *Condition for an increasing function to be Riemann-Stieltjes integrable.* (**d**: 4. >**4.3:5**(a))

Prove that $f\in\mathcal{R}(\alpha)$ if f is increasing and is continuous at every point where α is discontinuous. (Suggestion: Combine the ideas of the proofs of Theorems 6.8 and 6.9, using **4.3:5**(a). Incidentally, the idea of the first displayed formula in the proof of Theorem 6.9 is that the numbers $\Delta\alpha_i$ are all small. You won't be able to get a precise formula like that one in your proof, but you will want to use the same idea.)

6.3:6. *A condition for mutual integrability of* α *and* β. (**d**: 4. >**6.2:1**)

Show that the three conditions of **6.2:1** are also equivalent to: (iv) For every $x\in[a,b)$, either $\alpha(x) = \alpha(x+)$ or $\beta(x) = \beta(x+)$, and for every $x\in(a,b]$, either $\alpha(x) = \alpha(x-)$ or $\beta(x) = \beta(x-)$. (Suggestion: assuming (iv) holds, let $M = (\alpha(b) + \beta(b)) - (\alpha(a) + \beta(a))$. Given ε, find a partition such that in every interval of the partition, either $\Delta\alpha_i$ or $\Delta\beta_i$ is $< \varepsilon/2M$. Deduce that $\Delta\alpha_i\,\Delta\beta_i \leq (\Delta\alpha_i + \Delta\beta_i)(\varepsilon/2M)$, and sum these inequalities.)

6.3:7. *''Since* ε *is arbitrary ...''* (**d**: 2)

(a) Find the fallacy in the following argument:

''Theorem'' *Every function* $f: X \to Y$ *between metric spaces is continuous.*

Proof Given any $\varepsilon > 0$ and any $x\in X$, take $\delta = 1$, and consider a point $y \neq x$ of X satisfying $d(x,y) < \delta$. Choose any $C > d(f(x), f(y))/\varepsilon$. Multiplying this inequality by ε, we get

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Answer to True/False question **6.3:0.** (a) F.

$$d(f(x), f(y)) < C\varepsilon.$$

But since ε was arbitrary, we can choose it to make this as small as we like, proving continuity.

(b) The ends of the proofs of Theorem 3.50 (pp.74-75) and Theorem 6.11 (p.127) use a similar argument that since ε is arbitrary, an expression having ε as a factor can be made arbitrarily small, yielding a desired conclusion. Why do those proofs not suffer from the same fallacy as above?

6.4. Basic properties (beginning of Rudin's section **PROPERTIES OF THE INTEGRAL).** (pp.128-130)

Relevant exercises in Rudin:

6:R5. *Does $f^2 \in \mathscr{R}$ or $f^3 \in \mathscr{R}$ imply $f \in \mathscr{R}$?* (**d**:2)

6:R10. *Hölder's inequality.* (**d**:3,2,4,3)

The only way I can see to do part (*a*) is quite roundabout: Let $u^p = s$, $v^q = t$, $1/p = a$, and hence, by the first displayed equation, $1/q = 1 - a$. Turn the resulting inequality into an inequality concerning s/t, and write $s/t = r$. The resulting inequality will be true for $r = 1$; prove it for general r using the Mean Value Theorem, assuming the standard formula for the derivative of x^a. (Without the above hint, I would rate that part at least **d**:4.)

Hint for part (*c*) in the case where neither of the integrals on the right is zero: Use a scalar multiplication to reduce to the case where those integrals are 1, and apply (*b*).

I don't know what method Rudin had in mind for the case where one or both of those integrals is zero. It could be proved using results from Chapter 11, but these are not available yet. One method that will work is to approximate f and/or g by functions for which the integrals in question are nonzero, and show that the inequality for those approximating functions implies the same inequality for the limit functions. Another proof of this case of (*c*) for the special case $p = q = 2$ is given in **6.4:2** below, and an alternative development of the rest of part (*c*) for those values of p and q in **6.4:3**.

Part (*d*), of course, depends on **6:R7** and **6:R8**. (I didn't indicate this above because parts (*a*)-(*c*), which form the core of the exercise, do not.)

6:R11. *The triangle inequality for the L^2 norm.* (**d**:1. >**6:R10**(*c*), or **6.4:2** and **6.4:3**)

The ''Schwarz inequality'' that Rudin refers to here is not the result of that name which we saw in Theorem 1.35, but the version of that result with integration replacing summation referred to in part (*c*) of the preceding exercise. As the difficulty-rating indicates, this exercise is easy – assuming the difficult exercise **6:R10** discussed above, or the somewhat easier substitute exercises given below. Incidentally, $\|u\|_2$ is known as ''the L^2 norm of u''; hence the titles I have given this and the next exercise.

6:R12. *An integrable function is L^2-approximable by a continuous function.* (**d**:3. >**6:R11**)

Exercises not in Rudin:

6.4:0. *Say whether the following statement is true or false.*

(a) If $f \in \mathscr{R}(\alpha)$ on an interval $[a, b]$, then $f \in \mathscr{R}(\alpha)$ on every subinterval $[c, d] \subseteq [a, b]$.

6.4:1. *Integration with respect to α and α^{-1}.* (**d**:1)

A real-valued function α on a set E of real numbers is said to be *strictly increasing* if for all $x, y \in E$, $x < y \Rightarrow \alpha(x) < \alpha(y)$. Clearly, such a function is one-to-one. If α is a strictly increasing *continuous* function on an interval $[a, b]$, the Intermediate Value Theorem shows that α is onto $[\alpha(a), \alpha(b)]$, hence is a bijection from $[a, b]$ to $[\alpha(a), \alpha(b)]$, hence it has an inverse function, α^{-1}: $[\alpha(a), \alpha(b)] \to [a, b]$.

For such an α, show that for every real-valued function f on $[a, b]$ we have

$$\int_a^b f(x)\, d\alpha(x) = \int_{\alpha(a)}^{\alpha(b)} f(\alpha^{-1}(x))\, dx,$$

in the sense that if either side is defined, so is the other, and they are then equal.

(This result is related to Theorem 6.19; but that is outside of section 6.4, so you can't use it here.)

6.4:2. *Functions that behave like zero under Riemann-Stieltjes integration*. (**d**:3)

(a) Suppose that $f \in \mathcal{R}(\alpha)$ on $[a,b]$, and that $\int_a^b f(x)^2\, d\alpha(x) = 0$. Prove that for all $g \in \mathcal{R}(\alpha)$ one also has $\int_a^b f(x)\, g(x)\, d\alpha(x) = 0$.

Suggestion: Fixing g, verify on general principles that for all real numbers t, $\int_a^b (f(x) + t\, g(x))^2\, d\alpha(x) \geq 0$, and that as a function of t, this integral has a local minimum at $t = 0$. Now expand the integral in terms of the integrals of f^2, fg and g^2, and draw a conclusion.

(b) Deduce that if $f \in \mathcal{R}(\alpha)$ on $[a,b]$, and $f(x) \geq 0$ for all x, then the following conditions are equivalent: (i) $\int_a^b f(x)\, d\alpha(x) = 0$. (ii) $\int_a^b f(x)^2\, d\alpha(x) = 0$. (iii) $\int_a^b f(x)\, g(x)\, d\alpha(x) = 0$ for all $g \in \mathcal{R}(\alpha)$. (Hint: Can you write f as the square of a function in $\mathcal{R}(\alpha)$?)

(c) Show that if from part (b) we delete the assumption that $f(x) \geq 0$ for all x, then the equivalence still holds, with $f(x)$ replaced by $|f(x)|$ in statement (i), but not in statements (ii) and (iii). (Hint: $f = (|f|+f)/2 - (|f|-f)/2$.)

6.4:3. *The Schwarz inequality for integrals*. (**d**:3. >**6.4:2**)

The calculations by which we proved the Schwarz inequality for n-tuples of real numbers in **1.7:2** can be mimicked using integrals instead of sums. Given an increasing function α, the analog of the dot product for $f, g \in \mathcal{R}(\alpha)$ is $\int_a^b fg\, d\alpha$. Use the method of that exercise to obtain a Schwarz inequality for such functions (namely, the bottom display on p.139 with $p = q = 2$), assuming the integral of the square of each function is nonzero.

For vectors, the case of ''zero norm'' created no difficulty, because vectors of zero norm were zero; however, the analogous statement for integrals is not true (cf. **6:R1**, p.138). Show, however, that that case can be handled with the help of **6.4:2** above.

6.4:4. *Description of* $\mathcal{R}(\alpha+\beta)$. (**d**:2)

Show that if α and β are increasing functions on $[a,b]$, then $\mathcal{R}(\alpha+\beta) = \mathcal{R}(\alpha) \cap \mathcal{R}(\beta)$.

6.4:5. *Extending the Riemann-Stieltjes integral to the case of non-increasing* α. (**d**:3)

The definition of the Riemann-Stieltjes integral $\int f\, d\alpha$ requires that α be an increasing function. In this exercise we will see that, having defined such integrals for increasing α, we can extend the definition in a natural way to the wider class of all functions that can be written as differences of increasing functions. (Exercise **5.2:3** looked at that class of functions; but the result proved there is not needed for this exercise.)

Suppose that f and λ are functions on $[a,b]$, and that we can express λ as

$$\lambda = \alpha_1 - \alpha_2,$$

where α_1 and α_2 are increasing functions, and f is integrable with respect to both α_1 and α_2. Show that if we define

$$\int f d\lambda = \int f d\alpha_1 - \int f d\alpha_2$$

then $\int f d\lambda$ is well-defined, independent of our choice of decomposition of λ as $\alpha_1 - \alpha_2$.

Thus, what you must prove is that if λ can also be written $\lambda = \beta_1 - \beta_2$, where β_1 and β_2 are increasing and f is integrable with respect to both β_1 and β_2, then

$$\int f d\alpha_1 - \int f d\alpha_2 = \int f d\beta_1 - \int f d\beta_2.$$

(Suggestion: abbreviating $\int f d\alpha_1 - \int f d\alpha_2$ to $I(\alpha_1, \alpha_2)$, show that $I(\alpha_1, \alpha_2) = I(\alpha_1 + \beta_2, \alpha_2 + \beta_2)$ $= I(\alpha_2 + \beta_1, \alpha_2 + \beta_2) = I(\beta_1, \beta_2)$.)

6.4:6. *Converse to Theorem 6.12(c)*. (**d**:2)

Show that if $a < c < b$, and α is a monotonically increasing function on $[a,b]$, and f is a function on $[a,b]$ such that $f \in \mathcal{R}(\alpha)$ both on $[a,c]$ and on $[c,b]$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Thus the equality

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha,$$

Answer to True/False question **6.4:0.** (a) T.

which Rudin proves under the hypotheses of Theorem 6.12(c), also holds in this situation.

6.4:7. *Integrable functions can have infinitely many big jumps.* (**d**: 1,3)

Let $s_1 < s_2 < \ldots < s_n < \ldots$ be elements of an interval $[a,b]$ such that $s_1 = a$ and $\lim_{n\to\infty} s_n = b$. Let $f: [a,b] \to [0,1]$ be defined by the condition that for $x \in [s_i, s_{i+1})$, $f(x) = (x - s_i)/(s_{i+1} - s_i)$, while $f(b) = 0$.

(a) At what points x is f discontinuous, and at each of these points, what are the values of $f(x)$, $f(x-)$, and $f(x+)$ if the latter two are defined? (No proof required for this part.)

(b) Assuming the properties noted in part (a), show that $f \in \mathscr{R}$.

The above example is generalized in the next exercise.

6.4:8. *Integrability piece by piece.* (**d**: 3)

Let $s_1 < s_2 < \ldots < s_n < \ldots$ be elements of an interval $[a,b]$ such that $s_1 = a$ and $\lim_{n\to\infty} s_n = b$. Let α be an increasing function on $[a,b]$, and let f be any bounded real-valued function on $[a,b]$. For each positive integer i, let f_i denote the restriction of f to $[s_i, s_{i+1}]$, and α_i the restriction of α to that interval.

Prove that $f \in \mathscr{R}(\alpha)$ if and only if for every i, $f_i \in \mathscr{R}(\alpha_i)$, and that when this holds, we have

$$\int_a^b f\,d\alpha \;=\; \Sigma_{i=1}^{\infty} \left(\int_{s_i}^{s_{i+1}} f\,d\alpha\right),$$

with the sum on the right absolutely convergent. (The ith summand of that sum could, of course, be written more formally as $\int_{s_i}^{s_{i+1}} f_i\,d\alpha_i$.)

6.4:9. *Functions with only countably many discontinuities are integrable.* (**d**: 2,3)

Let us begin by proving explicitly an ''intuitively obvious'' general fact that we will need.

(a) Suppose an interval $[a,b]$ is covered by finitely many open neighborhoods, $N_{\varepsilon_1}(p_1), \ldots, N_{\varepsilon_m}(p_m)$. Show that there is a partition $x_0, \ldots, x_n$ of $[a,b]$ such that every subinterval $[x_{i-1}, x_i]$ is contained in (at least) one of the neighborhoods $N_{\varepsilon_j}(p_j)$. Show further that in this situation, given any subset $S \subseteq \{1, \ldots, m\}$, if we let T denote the set of indices $i \in \{1, \ldots, n\}$ such that $[x_{i-1}, x_i]$ is contained in $N_{\varepsilon_1}(p_j)$ for some $j \in S$, then $\Sigma_{i\in T}\,\Delta x_i \leq \Sigma_{j\in S}\, 2\varepsilon_j$.

(Suggestion: Let $y_1 < \ldots < y_k$ be those boundary points of the neighborhoods $N_{\varepsilon_i}(p_i)$ – i.e., points of the form $p_i - \varepsilon_i$ or $p_i + \varepsilon_i$ – that lie in (a,b), arranged in increasing order, and let $y_0 = a$, $y_{k+1} = b$. Use the partition $y_0 < (y_0 + y_1)/2 < (y_1 + y_2)/2 < \ldots < (y_{k-1} + y_k)/2 < (y_k + y_{k+1})/2 < y_{k+1}$, verifying that it has the required properties.)

Now for the interesting result.

(b) Show that a bounded real-valued function f on an interval $[a,b] \to R$ which is continuous except at a countable set of points is Riemann-integrable.

(Suggestion: Suppose f continuous at all points except $s_1, s_2, \ldots s_i, \ldots$. Given ε, choose a series $\Sigma\,\delta_i$ of positive real numbers which converges to a sum $< \varepsilon$. Surround each point s_i by the neighborhood $N_{\delta_i}(s_i)$, while for each point x where f is continuous, show that there is a neighborhood $N_{\delta(x)}(x)$ such that $\mathrm{diam}(f(N_{\delta(x)}(x))) < \varepsilon$. These two sorts of neighborhoods together cover $[a,b]$. Take a finite subcovering, choose a partition as in part (a), deduce that the total length of the intervals $[x_{i-1}, x_i]$ with $\mathrm{diam}(f([x_{i-1}, x_i])) > \varepsilon$ is small, and complete the proof like that of Theorem 6.11.)

6.4:10. *A Riemann-integrable function with uncountably many discontinuities.* (**d**: 3)

Let $f: [0,1] \to R$ denote the function which has the value 1 at all points of the Cantor set, and 0 elsewhere. Show that f is discontinuous at uncountably many points, but is Riemann integrable. Determine its integral.

(Suggestion: Consider the partition of $[0,1]$ into 3^n equal intervals, and count the number of such intervals which contain at least one point of the Cantor set. The fact that many of them have just an endpoint in the Cantor set is a minor nuisance, but even counting those, the fraction of intervals containing a point of the Cantor Set behaves well as $n \to \infty$.)

6.4:11. *One case of Theorem 6.12(a).* (**d**: 1)

Prove the last equation in Theorem 6.12(a) in the case where $c < 0$.

(Not hard, but one needs to notice that in proving that equation, the cases $c > 0$ and $c < 0$ must be distinguished.)

6.4:12. *Theorem 6.11 for a function φ of several variables.* (**d**: 3)

Suppose α is a monotonically increasing real-valued function on an interval $[a,b]$, and $f_1, \ldots, f_k$ are real-valued functions on $[a,b]$ which each belong to $\mathscr{R}(\alpha)$. Let us write $\mathbf{f}: [a,b] \to R^k$ for the function defined by $\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$. Let K be any compact subset of R^k containing $\mathbf{f}([a,b]) = \{\mathbf{f}(x) \mid x \in [a,b]\}$, and let φ be any continuous real-valued function on K.

Below, you will prove that the function $\varphi \circ \mathbf{f} : [a,b] \to R$ defined by $(\varphi \circ \mathbf{f})(x) = \varphi(\mathbf{f}(x))$ also belongs to $\mathscr{R}(\alpha)$, generalizing Rudin's Theorem 6.11.

(a) Given $\varepsilon > 0$, show that there is a $\delta > 0$ such that for $p, q \in K$ one has $d(p,q) < \delta \Rightarrow |\varphi(p) - \varphi(q)| < \varepsilon$, and a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that for $j = 1, \ldots, k$, $U(P,\alpha,f_j) - L(P,\alpha,f_j) < \delta\,\varepsilon$.

(b) For δ and $P = \{x_0, \ldots, x_n\}$ as above, let $A \subseteq \{1, \ldots, n\}$ be the set of indices such that for $j = 1, \ldots, k$,

$$\sup_{x \in [x_{i-1}, x_i]} f_j(x) \ - \ \inf_{x \in [x_{i-1}, x_i]} f_j(x) \ < \ \delta/\surd k.$$

Show that for $i \in A$ and $x, y \in [x_{i-1}, x_i]$, one has $d(\mathbf{f}(x), \mathbf{f}(y)) < \delta$. Deduce from this an upper bound on the sum of the terms of $U(P, \alpha, \varphi \circ \mathbf{f}) - L(P, \alpha, \varphi \circ \mathbf{f})$ indexed by members of A, such that this upper bound approaches 0 as $\varepsilon \to 0$.

(c) Let $B = \{1, \ldots, n\} - A$. Prove an upper bound on $\Sigma_{i \in B} \Delta\alpha_i$ that approaches 0 as $\varepsilon \to 0$. (Hint: ε was used in choosing P. Note that k, $\mathbf{f}$ and φ are fixed in this exercise, so the bound can depend on these.)

(d) Deduce a bound on $U(P, \alpha, \varphi \circ \mathbf{f}) - L(P, \alpha, \varphi \circ \mathbf{f})$ that approaches 0 as $\varepsilon \to 0$, and conclude that $\varphi \circ \mathbf{f} \in \mathscr{R}(\alpha)$, as claimed.

Remark: Rudin proves Theorem 6.11 only for functions of one variable. He deduces Theorem 6.13(a), that a product of functions in $\mathscr{R}(\alpha)$ again lies in $\mathscr{R}(\alpha)$, by a trick that reduces the two-variable multiplication operation to the one-variable squaring operation. The trick is impressive, but that result suggests the question of whether a similar result holds for continuous functions other than multiplication. The above exercise answers this question.

6.5. Step functions, differentiable α, and change of variables (end of Rudin's section **PROPERTIES OF THE INTEGRAL**). (pp.130-133)

Relevant exercises in Rudin: None

Exercises not in Rudin:

6.5:0. *Say whether each of the following statements is true or false.*

(a) There exists an increasing function α on $[0,1]$ such that for every continuous function f on that interval, $\int_0^1 f\,d\alpha = \Sigma\ 2^{-n} f(1/n)$.

(b) There exists an increasing function α on $[0,1]$ such that for every continuous function f on that interval, $\int_0^1 f\,d\alpha = \Sigma\ (1/n)\, f(2^{-n})$.

(c) If α increases monotonically on $[a,b]$ and is differentiable, then $\alpha' \in \mathscr{R}$ on $[a,b]$.

(d) If $f \in \mathscr{R}$ on $[0,1]$, then $\int_0^1 f(x)\,dx = \int_0^{1/2} f(2x)\,d(2x)$.

6.5:1. *Getting the Fundamental Theorem of Calculus from Theorem 6.17.* (**d**: 2. >**6.3:1**)

This exercise obtains of one of the main results of the next section from a result in this section.

Let $f \in \mathscr{R}$ on $[a,b]$, and suppose there exists a differentiable function F on $[a,b]$ such that $F' = f$.

(a) Show that if $f(x) \geq 0$ for all $x \in [a,b]$, then one can apply Theorem 6.17 with the constant function

1 as the ``f'' of that theorem, and F as the α of that theorem, and that **6.3:1** above is applicable to one side of the formula you get. Show the formula that results from applying **6.3:1** to that side.

(b) Deduce that the same formula holds if f can be written $f_1 - f_2$, where each of f_1 and f_2 is nonnegative, is Riemann-integrable, and is the derivative of a differentiable function.

(c) Show that any f which is Riemann-integrable and is the derivative of a differentiable function can in fact be written $f = f_1 - f_2$ as in (b). (Hint: Take f_2 to be an appropriate constant function.) Conclude that the formula you got in (a) is applicable to any such f.

6.5:2. *Strengthening Theorem 6.16.* (**d**:2)

Show that Theorem 6.16 (p.130) remains valid if the assumption that f is continuous on $[a,b]$ is replaced by the condition that it is bounded, and is continuous at each of the points s_n. Specifically –

We see that no continuity condition is needed until display (24). Say why that display is valid under the condition stated above. Then modify the remainder of the proof (in which Rudin uses integrability of f with respect to α_2, which under his assumptions follows from Theorem 6.8) to conclude that (23) holds with the upper, respectively the lower integral in place of the left-hand side. Conclude that $f \in \mathscr{R}(\alpha)$ and that (23) holds.

6.5:3. *Weaker conditions for integrability.* (**d**:3. >**6.5:2**)

Let $J(x)$ be defined like $I(x)$ at the bottom of p.129, except that $J(0) = 1$. (Equivalently, $J(x) = 1 - I(-x)$ for all x.)

(a) Show that every increasing function $\alpha(x)$ on an interval $[a,b]$ can be written

$$\alpha(x) \;=\; \Sigma_1^\infty\, c_n\, I(x - s_n) \;+\; \Sigma_1^\infty\, d_n\, J(x - s_n) \;+\; \beta(x),$$

where $\Sigma\, c_n$ and $\Sigma\, d_n$ are convergent series of nonnegative real numbers, (s_n) is a sequence of distinct points of $[a,b]$, and β is a *continuous* increasing function on $[a,b]$.

(Hint: If α is discontinuous at infinitely many points, can you write the set of these points as $\{s_n \mid n = 1, 2, \ldots\}$? If so, what should the c_n and d_n be to make the above equation plausible? Once you have found these, to prove that $\alpha(x) - \Sigma\, c_n\, I(x - s_n) - \Sigma\, d_n\, J(x - s_n)$ is increasing, approximate this function using finite partial sums in place of the infinite sums, and prove each of the resulting functions increasing. Finally, verify continuity.)

(b) Taking for granted that the version of Theorem 6.16 described in **6.5:2** is also valid with J in place of I, deduce that Theorem 6.9 remains valid if the assumption that α is continuous on $[a,b]$ is replaced by the assumption that it is continuous at every point where f is discontinuous.

(The proof of the result gotten by replacing ``$I$'' with ``J'' in **6.5:2** is virtually identical to the original proof of that exercise. It can also be gotten using the next exercise.)

6.5:4. *Integration with respect to* $d(-\alpha(-x))$. (**d**:2)

Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$. We would like to be able to apply Theorem 6.19 in the case $\varphi(x) = -x$, and deduce that $\int_{-a}^{-b} f(-x)\, d\alpha(-x) = \int_a^b f(x)\, d\alpha(x)$. Unfortunately, $\alpha(-x)$ is not an increasing function of x, and $-a$ is not $< -b$, so the left-hand integral has not been defined. However, $-\alpha(-x)$ is an increasing function, so

(a) Show that

$$\int_{-b}^{-a} f(-x)\, d(-\alpha(-x)) \;=\; \int_a^b f(x)\, d\alpha(x).$$

(Remark: Rudin doesn't like showing the variable in integrations. To satisfy his preference, we could let $\rho : R \to R$ (ρ = rho, for ``reversal'') be defined by $\rho(x) = -x$, and write the above equation $\int (f \circ \rho)\, d(\rho \circ \alpha \circ \rho) = \int f\, d\alpha$.)

(b) From part (a) and Theorem 6.19, deduce that if, in the first line of that theorem we change ``increasing'' to ``decreasing'', then we get a formula like (32), but with $d\beta$ changed to $d(-\beta)$.

Answers to True/False question **6.5:0.** (a) T. (b) F. (c) F. (d) T.

6.6. INTEGRATION AND DIFFERENTIATION (the Fundamental Theorem of Calculus). (pp.133-134)

Relevant exercises in Rudin:

6:R9. *Integration by parts for improper integrals.* (**d**:4. uses definitions in **6:R7**,**6:R8**)

6:R13. *Calculations involving* $\int \sin(t^2)\,dt$. (**d**:3)

Note that in this exercise, parentheses ( ) and brackets [ ] do *not* denote the ''fractional part'' and ''integer part'' functions of **4:R16**. They are simply being used to make very clear what the squaring operation is and isn't being applied to.

6:R14. *Calculations involving* $\int \sin(e^t)\,dt$. (**d**:3. >**6:R13**)

6:R15. *Properties of a function such that* $\int f^2 dt = 1$. (**d**:2. >**6:R10**(c) or **6.4:3**)

6:R16. *The Riemann zeta-function.* (**d**:3. uses definitions in **6:R8**)

Before attempting this, you should be sure you are familiar with the greatest-integer function $[x]$; e.g., sketch its graph. The improper integrals are to be understood in the sense of **6:R8**, p.138 (but the result of that exercise isn't needed for this one). To evaluate the corresponding integrals on intervals $[1,N]$ as suggested in Rudin's hint, compute them for the subintervals where $[x]$ is constant (except at the endpoints), and use Theorem 6.12(b). In doing the integrations, you can assume the formula for the derivative of x^c, but justify the *integration* formula you get from that formula using a theorem in this chapter; also justify the way you handle the behavior at endpoints of intervals of integration (hint: **6:R1**). In the final step of deducing from the values of the integral over intervals $[1,N]$ the value of the improper integral, remember that you need a limit of $\int_1^C ...$ as C approaches $+\infty$ through all real values, not just through the integer values examined so far.

Get (b) from (a). (The point of (b) is that it gives a formula for the zeta function that makes sense not only for $s\in(1,+\infty)$, where the original series converges, but also for $s\in(0,1)$.)

6:R17. *Integration by parts for Riemann-Stieltjes integrals.* (**d**:4)

Rudin's *Hint* begins ''Take g real, without loss of generality.'' What this means is, prove the result for an R-valued function g, then deduce from that the corresponding statement for an R^k-valued function $\mathbf{g}$.

In obtaining the relation Rudin gives at the end of his *Hint*, you might use Theorem 3.41, or **3.11:1** above.

Exercises not in Rudin:

6.6:0. *Say whether each of the following statements is true or false.*

(a) If $f\in\mathscr{R}$ on $[a,b]$, and for all $x\in[a,b]$ we define $F(x) = \int_a^x f(t)\,dt$, then F is differentiable and $F' = f$.

(b) If f is a continuous function on $[a,b]$, and for all $x\in[a,b]$ we define $F(x) = \int_x^b f(t)\,dt$, then F is differentiable and $F' = -f$.

6.6:1. *A sort of derivative formula for Riemann-Stieltjes integrals.* (**d**:2)

On $[a,b]$, let α be a strictly increasing function (as defined in **5.2:1**(b)) and f a continuous function, and for $x\in[a,b]$ define $F(x) = \int_a^x f(t)\,d\alpha(t)$. Show that for all $x\in[a,b]$, $dF(x)/d\alpha(x) = f(x)$, where the left-hand side is defined as $\lim_{t\to x}(F(x)-F(t))/(\alpha(x)-\alpha(t))$, and the equality includes the assertion that this limit exists.

6.6:2. *Repeated integration reduces to a single integration.* (**d**:3,4,4)

(a) Show that if f is a continuous function on $[a, b]$, then

$$\int_{t=a}^{b}\left(\int_{s=a}^{t} f(s)\,ds\right)dt \;=\; \int_{t=a}^{b}(b-t)f(t)\,dt.$$

Hint: Rewrite the above equation with arbitrary $x\in[a,b]$ in place of b, and name the left-hand side $P(x)$ and the right-hand side $Q(x)$. Show that both P and Q are differentiable functions of x, and have the same derivatives. In figuring out how to differentiate Q, use Theorem 6.12.

One can also get this formula using results about change of order of integration; but Rudin will not treat that subject till Chapter 10.

(b) Find a similar formula for the n-fold iterated integral of f, of which the above is the $n=2$ case.

(c) Show that the result of (a) and, if you did it, (b), continues to hold if f is merely assumed Riemann-integrable, but not necessarily continuous. (Hints: To show the right-hand side is differentiable with the correct derivative, first verify that if a function $u(x)$ is bounded, then for any x_0, the function $(x-x_0)\,u(x)$ is continuous at x_0, while if $u(x)$ is continuous at x_0, then $(x-x_0)\,u(x)$ is differentiable at x_0. Now suppose you want to find the derivative of the right-hand integral at x_0. Rewrite the factor $(x-t)$ as $(x-x_0)+(x_0-t)$, and treat each of the resulting integrals in the neighborhood of x_0 with the help of the above results.)

6.6:3. *The Fundamental Theorem, minus the condition that the derivative be integrable.* (**d**: 2)

Prove the following generalization of Theorem 6.21, p.134: If F is any differentiable function on $[a,b]$, then

$$\underline{\int}_a^b F'(x)\,dx \;\le\; F(b)-F(a) \;\le\; \overline{\int}_a^b F'(x)\,dx.$$

6.6:4. *A formal product law for* $d(\alpha\beta)$. (**d**: 3)

Let f be a function on $[a,b]$ and α, β monotonically increasing nonnegative functions on $[a,b]$ such that $f\in\mathscr{R}(\alpha)\cap\mathscr{R}(\beta)$, $\alpha\in\mathscr{R}(\beta)$, and $\beta\in\mathscr{R}(\alpha)$. (By **6.2:1**, these conditions are slightly redundant.) Prove that

$$\int f\,d(\alpha\beta) \;=\; \int f\,\alpha\,d\beta \;+\; \int f\,\beta\,d\alpha.$$

(If we use this as a formula for evaluating $\int f\,\alpha\,d\beta$ in terms of the other two integrals, it can be thought of as a generalization of integration by parts.)

6.6:5. *Repeated Riemann-Stieltjes integration.* (**d**: 4)

We shall obtain here a generalization of the result of **6.6:2** in which ds and dt are replaced by more general expressions $d\alpha(s)$ and $d\beta(t)$.

(a) Suppose α, β are increasing functions on $[a,b]$ such that $\alpha\in\mathscr{R}(\beta)$, and $f\in\mathscr{R}(\alpha)\cap\mathscr{R}(\beta)$. Show that $\int_a^x f(t)\,d\alpha(t)$, regarded as a function of x, belongs to $\mathscr{R}(\beta)$, and that

$$\int_{t=a}^{b}\left(\int_{s=a}^{t} f(s)\,d\alpha(s)\right)d\beta(t) \;=\; \int_{t=a}^{b}(\beta(b)-\beta(t))\,f(t)\,d\alpha(t).$$

(Suggestion: First consider the case where f is everywhere >0, since in this case one can easily describe the least and greatest values of $\int_{s=a}^{t} f(s)\,d\alpha(s)$ on any interval as the values at the respective ends of the interval; then get the general case by writing any $f\in\mathscr{R}(\alpha)\cap\mathscr{R}(\beta)$ as a difference of two nonnegative-valued functions in $\mathscr{R}(\alpha)\cap\mathscr{R}(\beta)$. Cf. the method of passing from part **6.5:1**(b) to (c).)

(b) Show that given continuous functions u and v on $[a,b]$, there exists a continuous function w of two variables such that for any $f\in\mathscr{R}$ on $[a,b]$, one has

$$\int_{t=a}^{x} u(t)\left(\int_{s=a}^{t} v(s)\,f(s)\,ds\right)dt \;=\; \int_{t=a}^{x} w(x,t)\,f(t)\,dt.$$

Hint: Prove the case where u and v are nonnegative-valued using (a).

6.6:6. *A characterization of the Riemann-Stieltjes integral.* (**d**: 2, 4, 4)

Let α be an increasing function on $[a,b]$, and f any bounded function on that interval. Given x_1, x_2 with $a\le x_1<x_2\le b$, let us use ΔF as an abbreviation for $F(x_2)-F(x_1)$ and $\Delta\alpha$ as an abbreviation for $\alpha(x_2)-\alpha(x_1)$.

(a) Show that if $f\in\mathscr{R}(\alpha)$ and we define $F(x)=\int_{t=a}^{x} f(t)\,d\alpha(t)$, then

(i) $F(a)=0$,

(ii) For all x_1, x_2 with $a\le x_1<x_2\le b$, we have $\Delta F\in[(\Delta\alpha)(\inf f(x)),\,(\Delta\alpha)(\sup f(x))]$, where the $\inf$ and $\sup$ are over all $x\in[x_1,x_2]$.

Answers to True/False question **6.6:0.** (a) F. (b) T.

(We could write the conclusion of (ii) more suggestively as $\Delta F/\Delta\alpha \in [\inf f(x), \sup f(x)]$, were it not that $\Delta\alpha$ might be zero for some choices of x_1 and x_2.)

(b) Show that F is the unique function satisfying (i) and (ii).

(c) Show that for any bounded function f on $[a,b]$, if there is a unique function F on $[a,b]$ satisfying these conditions, then $f \in \mathscr{R}(\alpha)$.

Suggestion: Show that for any bounded f, the upper and lower integrals $F_+(x) = \overline{\int}_{t=a}^{x} f(t)\,d\alpha(t)$ and $F_-(x) = \underline{\int}_{t=a}^{x} f(t)\,d\alpha(t)$ each satisfy the indicated conditions, and that any function F satisfying those conditions satisfies $F_-(x) \le F(x) \le F_+(x)$ for all $x \in [a,b]$.

6.6:7. *A change-of-variables result.* (**d**: 3. >**6.6:6**)

Suppose α is a monotonically increasing real-valued function on $[a, b]$ and f, g are continuous real-valued functions on that interval, with g nonnegative-valued. Prove

$$\int_{t=a}^{x} f(t)\; d\Big(\int_{s=a}^{t} g(s)\,d\alpha(s)\Big) \;=\; \int_{t=a}^{x} f(t)\,g(t)\,d\alpha(t).$$

6.7. INTEGRATION OF VECTOR-VALUED FUNCTIONS. (pp.135-136)

Relevant exercises in Rudin: None

(But since the concepts of this section are essential to the next, exercises for the next section also test the material in this one.)

Exercises not in Rudin:

6.7:0. *Say whether each of the following statements is true or false.*

(a) If $\mathbf{f}$ is a differentiable R^k-valued function on $[a,b]$ and $\mathbf{f}' \in \mathscr{R}$, then $\int_a^b \mathbf{f}'\,dx = \mathbf{f}(b) - \mathbf{f}(a)$.

(b) If α is a monotonically increasing function on $[a,b]$ and $\mathbf{f}$ is a differentiable R^k-valued function on $[a,b]$ such that $\mathbf{f}' \in \mathscr{R}(\alpha)$, then $\left|\int_a^b \mathbf{f}'\,d\alpha\right| \ge \int_a^b |\mathbf{f}'|\,d\alpha$.

6.7:1. *Integrability of vector-valued functions expressed in terms of partitions.* (**d**: 2)

Let $\mathbf{f}$ be a function $[a,b] \to R^k$, and $\alpha\colon [a,b] \to R$ an increasing function.

(a) Show that $\mathbf{f}$ is integrable with respect to α if and only if for every $\varepsilon > 0$ there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that

$$\Sigma_{i=1}^{n}\ \mathrm{diam}(\mathbf{f}([x_{i-1}, x_i]))\,\Delta\alpha_i \;<\; \varepsilon.$$

where "diam" denotes the diameter of a set (Definition 3.9, p.52).

In parts (b) and (c) below, assume $\mathbf{f}$ is indeed integrable with respect to α, and let ε be a positive real number, and $P = \{x_0, \ldots, x_n\}$ a particular partition making the above inequality hold.

(b) Show that for every choice of points $t_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$, we have

$$\left|\,(\Sigma_{i=1}^{n}\,\mathbf{f}(t_i)\,\Delta\alpha_i) - (\int_a^b \mathbf{f}\,d\alpha)\,\right| \;<\; \varepsilon.$$

(c) Show that every refinement P^* of P also satisfies the inequality of (a), and hence the condition of (b).

6.8. RECTIFIABLE CURVES. (pp.136-137)

Relevant exercises in Rudin:

6:R18. *Rectifiability and length of a curve don't depend on its range alone ... .* (**d**: 1,1,3)

Here e^{ix} is defined as in Example 5.17. You may assume differentiability of the sine and cosine function, and the standard formulas for their derivatives. The definition Rudin gives for $\gamma_3(t)$ is undefined for $t = 0$; let $\gamma_3(0) = 1$. Before beginning the exercise, point out why this choice makes $\gamma_3(t)$ continuous at 0.

6:R19. *... but they are not affected by reparametrization.* (**d**: 2)

Exercises not in Rudin:

6.8:0. *Say whether each of the following statements is true or false.*

(a) If γ is a rectifiable curve in R^k, then γ is differentiable and γ' is continuous.

(b) If γ is the curve in the complex plane C defined by $\gamma(t) = \cos t + i \sin t$, with the parameter interval $[0, 4\pi]$, then $\Lambda(\gamma) = 4\pi$.

6.8:1. *A space-filling curve can't be rectifiable.* (**d**:4)

In **7:R14** (p.168) Rudin will construct a ''space-filling curve''; specifically, a curve in R^2 whose image is the whole unit square $[0,1]^2$. Show that a curve with this property cannot be rectifiable.

One possible approach: Find a constant c such that every curve in R^2 whose image contains all points of a solid square of side s must have a sub-curve of length at least cs lying in the interior of that square; then use the fact that the unit square can be broken into n^2 squares of side $1/n$. (A variant of this argument is indicated in the next exercise.)

6.8:2. *Busy-body curves have to be long.* (**d**:4)

Prove that for every positive constant C there exists an $\varepsilon > 0$ such that any curve in R^2 whose image has points at distance $\leq \varepsilon$ from each point of the unit square $[0,1]^2$ must have length $\geq C$.

Show how the result of the preceding exercise follows from this.

6.8:3. *Integrability of γ' is enough to prove the curve-length formula.* (**d**:2. >**6.7:1**(a))

Suppose $\gamma\colon [a,b] \to R^k$ is a differentiable curve such that $\gamma' \in \mathscr{R}$. Exercise **6.7:1**(a) tells us that given $\varepsilon > 0$ we can find a partition $P = \{x_0, \ldots, x_n\}$ such that

$$\Sigma_{i=1}^{n} \operatorname{diam}(\mathbf{f}([x_{i-1}, x_i]))\,\Delta x_i \ < \ \varepsilon.$$

(a) Show that if P is such a partition, and we take any point t_i in each interval $[x_{i-1}, x_i]$, then $\Sigma \left| \,|\gamma(x_i) - \gamma(x_{i-1})| - |\gamma'(t_i)|\,\Delta x_i \right| < \varepsilon$.

(b) Deduce that Theorem 6.27 remains true if the assumption ''γ' is continuous'' is weakened to ''$\gamma' \in \mathscr{R}$''.

Chapter 7. Sequences and series of functions.

7.1. DISCUSSION OF THE MAIN PROBLEM. (pp.143-147)

Relevant exercises in Rudin: None

Exercises not in Rudin:

7.1:0. *Say whether each of the following statements is true or false.*

(a) If (f_n) is the sequence of functions defined by $f_n(x) = x^n$ on the interval [0,1], then the limit function of the sequence (f_n) is continuous.

(b) If (f_n) is the sequence of functions defined by $f_n(x) = x^n$ on the interval [0,2], then the sequence (f_n) does not approach a limit function.

(c) If (f_n) is a sequence of real-valued functions on R which converges pointwise to a function f, and each f_n is continuous, then so is f.

(d) If a series Σf_n of real-valued functions on a metric space X converges pointwise, and if each f_n is uniformly continuous, then so is the function Σf_n.

7.1:1. *Is a pointwise limit of (strictly) increasing functions (strictly) increasing?* (**d**:2)

(a) If (f_n) is a sequence of real-valued functions on R which converges pointwise to a function f, and each f_n is monotonically increasing, must f also be monotonically increasing? Give a proof or a counterexample.

Answers to True/False question **6.7:0.** (a) T. (b) F.

(b) Same question with ''monotonically increasing'' replaced by ''strictly increasing'' (defined in **5.2:1**).

The answer to one of the above parts requires an ''obvious'' property of limits of sequences of real numbers. Rudin does not seem to state this property directly, but you can get it from Theorem 3.19 combined with Example 3.18(c).

7.1:2. *Double limits vs. iterated limits.* (**d**: 1, 2, 2, 2)

Suppose X is a metric space, p an element of X, and for all positive integers m and n, $s_{m,n}$ is an element of X. This exercise will compare various versions of the concept of the points $s_{m,n}$ approaching p as m and n *both* go to ∞.

(a) Show that the following conditions are equivalent:

(i) For every $\varepsilon > 0$ there exist positive integers M and N such that for all $m \geq M$ and $n \geq N$, one has $d(s_{m,n}, p) < \varepsilon$.

(ii) For every $\varepsilon > 0$ there exists a positive integer N such that for all $m \geq N$ and $n \geq N$, one has $d(s_{m,n}, p) < \varepsilon$.

When the above equivalent conditions hold, we shall write $\lim_{m,n\to\infty} s_{m,n} = p$. If $\lim_{m,n\to\infty} s_{m,n} = p$ for some $p \in X$, we shall say that $\lim_{m,n\to\infty} s_{m,n}$ exists.

(b) Show that if $\lim_{m,n\to\infty} s_{m,n} = p$, and if for each positive integer m the limit $\lim_{n\to\infty} s_{m,n}$ exists, then $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n}$ also exists and equals p.

Since the definition of $\lim_{m,n\to\infty} s_{m,n}$ is symmetric in m and n, (b) also shows that if $\lim_{m,n\to\infty} s_{m,n} = p$ and for each n, the limit $\lim_{m\to\infty} s_{m,n}$ exists then $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n}$ exists and equals p.

Hence if $\lim_{m,n\to\infty} s_{m,n}$ and $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n}$ and $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n}$ all exist, then they are all equal; though Rudin has shown us that the last two alone may exist and not be equal.

In the remaining two parts, give examples of systems of real numbers $s_{m,n}$ such that –

(c) $\lim_{m,n\to\infty} s_{m,n}$ exists, but $\lim_{m\to\infty} s_{m,n}$ does not exist for any n, nor $\lim_{n\to\infty} s_{m,n}$ for any m.

(d) $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n}$ and $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n}$ both exist, and are equal, but $\lim_{m,n\to\infty} s_{m,n}$ does not exist.

In (c) and (d), point out why your examples have the indicated properties. You do not have to give formal proofs.

7.1:3. *Iterated limits and diagonal limits.* (**d**: 2)

Suppose X is a metric space, and for all positive integers m and n we have an element $s_{m,n} \in X$, such that for each m, the limit $\lim_{n\to\infty} s_{m,n}$ exists, and such that these limits approach a limit, $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = p$.

Show that there exists a sequence of positive integers $N_1, N_2, \ldots$ such that for every sequence of positive integers $n_1, n_2, \ldots$ satisfying $n_m \geq N_m$ one has $\lim_{m\to\infty} s_{m,n_m} = p$.

Suggestion: For each m, choose N_m so that for all $r \geq N_m$ one has $d(s_{m,r}, \lim_{n\to\infty} s_{m,n}) < 1/m$ (noting why such an N_m exists).

7.1:4. *The ''1 at rationals'' function isn't a pointwise limit of continuous functions.* (**d**: 5)

In Example 7.4, Rudin showed that the function f on R which has value 1 at every rational number and 0 at every irrational is a *pointwise limit of pointwise limits* of continuous functions.

Show, however, that this function f is not itself a pointwise limit of any sequence (f_n) of continuous functions.

7.1:5. *Variants of the preceding exercise.* (**d**: 5, 5)

Suppose f is a function from a complete metric space X to a metric space Y, and suppose Y has points y_0, y_1 such that the subsets $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are both dense in Y.

Answers to True/False question **6.8:0.** (a) F. (b) T. **Answers** to True/False question **7.1:0.** (a) F. (b) T. (c) F. (d) F.

(a) Show that f is not a pointwise limit of continuous functions $X \to Y$.

(b) Show that in the above situation, if f maps every point of X either to y_0 or to y_1 (as in **7.1:4**), and if $f^{-1}(y_1)$ is countable, then f cannot even be expressed as a pointwise limit of functions $X \to Y$ that are continuous at all points of $f^{-1}(y_1)$; but that it *can* be expressed as a pointwise limit of functions $X \to Y$ that are continuous at all points of $f^{-1}(y_0)$.

7.2. UNIFORM CONVERGENCE. (pp.147-148)

Relevant exercises in Rudin:

7:R2. *Uniform convergence of two sequences generally implies uniform convergence of their sum and product ... .* (**d**: 1, 3)

This result is used in the proof of Theorem 7.29 (p.161), where Rudin says ''the convergence being uniform in each case''. The exercise really should be worded more precisely to say that if (f_n) and (g_n) converge uniformly to f and g respectively, then $(f_n + g_n)$ converges uniformly to $f + g$, etc..

7:R3. *... the exception being the product, if the functions are not assumed bounded.* (**d**: 2)

7:R4. *Convergence questions for a particular series.* (**d**: 3)

Consider only values of x in $[0, +\infty)$. (If we allowed negative values, we would have to exclude those of the form $-1/n^2$, where one of the summands has denominator 0.)

In answering a question such as ''On what intervals does it converge uniformly?'' you should give enough information so that for any interval $[a, b] \subseteq R$, your answer determines whether or not the series converges uniformly on $[a, b]$. And as usual, assertions must be proved.

7:R5. *Example showing that absolute convergence of a series does not imply uniform convergence.* (**d**: 2)

7:R6. *Example showing that uniform convergence of a series does not imply absolute convergence.* (**d**: 1)

A simpler example than that of this exercise is any series of *constant* functions, whose values converge non-absolutely.

Exercises not in Rudin:

7.2:0. *Say whether each of the following statements is true or false.*

(a) The sequence (f_n) of functions on $[-1, 1]$ defined by $f_n(x) = x/n$ converges uniformly to the function 0.

(b) The sequence (f_n) of functions on $[0, +\infty)$ defined by $f_n(x) = x/n$ converges uniformly to the function 0.

(c) The sequence (f_n) of functions on $[0, 1]$ defined by $f_n(x) = x^n$ converges uniformly.

(d) The sequence (f_n) of functions on $[0, 1)$ defined by $f_n(x) = x^n$ converges uniformly.

(e) The sequence (f_n) of functions on $[0, 1]$ defined by $f_n(x) = (x/2)^n$ converges uniformly.

(f) If a sequence (f_n) of real-valued functions on a set X converges uniformly, and E is a subset of X, then the restrictions of the f_n to E also converge uniformly.

(g) If (f_n) is a sequence of real-valued functions on a set X, and there is a subset E of X such that the restrictions of the f_n to E converge uniformly, then (f_n) converges uniformly.

(h) If a sequence (f_n) of real-valued functions on a set X converges uniformly, then every subsequence (f_{n_k}) also converges uniformly.

7.2:1. *On a finite set, uniform and pointwise convergence are the same.* (**d**: 1)

Suppose (f_n) is a sequence of complex-valued functions on a finite set E. Show that (f_n) converges pointwise if and only if it converges uniformly.

7.3. UNIFORM CONVERGENCE AND CONTINUITY. (pp.149-151)

Relevant exercises in Rudin:

7:R8. *Uniform convergence of sums of step functions.* (**d**: 1)

7:R9. *When* $\lim f_n(x_n) = f(x)$. (**d**: 2)

For ''a set E'' read ''a metric space E''.

In the last sentence Rudin asks whether ''the converse'' is true. Let us take this to mean the statement ''If (f_n) is a sequence of continuous functions on a metric space E, and f a continuous function on E, such that for every sequence of points (x_n) in E which approach a limit x, one has $\lim_{n\to\infty} f_n(x_n) = f(x)$, then (f_n) converges uniformly to f.''

7:R14. *A space-filling curve.* (**d**: 3)

I've given several exercises that asked you to draw conclusions assuming the existence of such a curve. Here at last is the construction!

Notes on the first sentence of the exercise: Rudin introduces a function f, but what he says does not completely determine that function. The displayed formula determines it on the intervals $[0, 1/3]$ and $[2/3, 1]$; the values on $(1/3, 2/3)$ can be filled in in any way that connects continuously with the values at $1/3$ and $2/3$, and satisfies $0 \le f(t) \le 1$. (Draw a picture if this helps. For simplicity, you might choose the graph of f on $[1/3, 2/3]$ to be a straight line-segment.) Once f is determined on $[0,1]$, the condition $f(t+2) = f(t)$ determines it on $[2,3]$, $[4,5]$, etc., but not on $(1,2)$, $(3,4)$, etc.. Again, its values on $(1,2)$ may be chosen in any way that connects continuously with the values at the endpoints and satisfies $0 \le f(t) \le 1$, and when this has been done, the rule $f(t+2) = f(t)$ determines it on $(3,4)$ etc., so f is determined on all of R.

The reason Rudin leaves the values on $(1/3, 2/3)$ and $(1,2)$ unspecified is that they will not affect the properties he is going to prove. The values on the intervals where he precisely specifies f, and the general properties he specifies, are all he will need.

In following Rudin's *Hint* for that problem, you don't have to prove the first sentence of that hint; simply note what the displayed equations represent.

7:R24. *An isometric embedding of any metric space in a complete metric space.* (**d**: 3)

Exercises not in Rudin:

7.3:0. *Say whether each of the following statements is true or false.*

(a) If (f_n) is a sequence of real-valued functions on a metric space X which converges uniformly to a function f, and if f is continuous, then at least one of the functions f_n is continuous.

(b) If (f_n) is a sequence of continuous real-valued functions on a compact metric space K which converges pointwise to a continuous function f, and if for each $x \in K$ and each n, $f_n(x) \le f_{n+1}(x)$, then $f_n \to f$ uniformly on K.

(c) The function f defined by $f(x) = \sin(x^2)$ belongs to $\mathscr{C}(R)$.

(d) If (f_n) is a sequence of continuous bounded functions on a compact metric space K which converges pointwise to a continuous bounded function f, then $f_n \to f$ in the metric space $\mathscr{C}(K)$.

(e) If (f_n) is a sequence of continuous bounded functions on a compact metric space K which converges uniformly to a function f, then $f_n \to f$ in the metric space $\mathscr{C}(K)$.

(f) If X is a metric space, then every Cauchy sequence in $\mathscr{C}(X)$ converges.

(g) If K is a compact metric space, then $\mathscr{C}(K)$ is compact.

7.3:1. *The amoeba meets uniform convergence.* (**d**: 2. >**4.2:6**, **7:R2**)

For each positive integer n, let $p_n : [0,1] \to R$ be the function so named in Exercise **4.2:6**, i.e., the probability that an amoeba that divides every hour will have at least one descendent after n hours,

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Answers to True/False question **7.2:0.** (a) T. (b) F. (c) F. (d) F. (e) T. (f) T. (g) F. (h) T.

expressed in terms of the probability that an amoeba survives for one hour.

(a) With the help of the result of **4.2:6**(a), show that for every n, p_n is given by a polynomial in t, and that if we write $p_\infty(t)$ for $\lim_{n\to\infty} p_n(t)$, the convergence of the polynomials $p_n(t)$ to the function $p_\infty(t)$ is uniform on $[0,1]$.

Deduce that the convergence of the polynomials $t p_n(t)$ to the function $t p_\infty(t)$ is likewise uniform on that set. The fact that $t p_\infty(t)$ is the limit of a uniformly convergent sequence of polynomial functions on $[0,1]$ is what we will use in the remaining parts.

(b) From the above statement and the result of **4.2:6**(b), deduce using a change of variables that the function $[-1,1]\to R$ taking t to $\max(0,t)$ is the limit of a uniformly convergent sequence of polynomial functions on that interval, and that the same is true of the function $[-1,1]\to R$ taking t to $\max(0,-t)$. Adding these functions, deduce that the same is true of the absolute value function on $[-1,1]$.

(c) Deduce that the same is true of the absolute value function on $[-a,a]$ for any positive real number a.

(d) Show that by subtracting constants from the polynomials arising in (c), one can represent the absolute value function on $[-a,a]$ as the uniform limit of a sequence of polynomials each of which has the value 0 at 0.

Remark: Rudin gets the result of (d) in another way in **7:R23**. As he notes, this allows one to save a good bit of work in the last section of Chapter 7. I hope the word-problem about amoebas has provided an entertaining journey to this useful fact.

7.3:2. *A version of Theorem 7.13 for functions between metric spaces.* (**d**: 2)

Suppose K is a compact metric space and (f_n) a sequence of continuous functions from K to a metric space Y, which converges pointwise to a continuous function f. Suppose further that for all $x\in K$ and all n, $d(f_{n+1}(x), f(x)) \leq d(f_n(x), f(x))$. Show that $f_n \to f$ uniformly on K. (Suggestion: Either adapt the method of proof of Theorem 7.13 (p.150), or use that theorem.)

Show that Theorem 7.13 is implied by the above result.

7.3:3. *If (f_n) converges uniformly to a bounded function, then most f_n have a common bound.* (**d**: 1)

Let (f_n) be a sequence of real- or complex-valued functions on a set E, which converges uniformly to a function f. Show that if f is bounded, then there exist a constant M and an integer N such that for all $n > N$ and all $x\in E$, $|f_n(x)| \leq M$.

Show by example, however, that there may be values of n for which f_n is unbounded.

7.3:4. *A convergent series of nonnegative functions on a compact space converges uniformly.* (**d**: 1)

Show that if (f_n) is a sequence of nonnegative-valued continuous functions on a compact metric space K, and the series Σf_n converges pointwise to a continuous function, then it converges uniformly. (Hint: This follows easily from a result in this section.)

7.3:5. *Subsets of $\mathscr{C}(E)$ determined by limit-conditions are closed.* (**d**: 3, 1)

(a) Show that $\{f\in\mathscr{C}(R) \mid \lim_{x\to+\infty} f(x) \text{ exists}\}$ is a closed subset of $\mathscr{C}(R)$.

(b) Suppose E is a subset of a metric space X and $p\in X$ is a limit point of E. Deduce from Theorem 7.11 that $\{f\in\mathscr{C}(E) \mid \lim_{x\to p} f(x) \text{ exists}\}$ is a closed subset of $\mathscr{C}(E)$.

7.3:6. *Uniform continuity, and continuity of the translation-map.* (**d**: 2)

Let $f\in\mathscr{C}(R)$, and for each $c\in R$, let f_c be defined by $f_c(x) = f(x+c)$. Show that the map $h: R\to\mathscr{C}(R)$ given by $h(c) = f_c$ is continuous if and only if f is uniformly continuous.

7.3:7. *Discontinuities of uniform limits of discontinuous functions.* (**d**: 2, 2, 2, 3)

Let (f_n) be a sequence of real- or complex-valued functions on R which converges uniformly to a function f.

(a) Show that if each f_n has at most countably many discontinuities, then the same is true of f.

Answers to True/False question **7.3:0.** (a) F. (b) T. (c) T. (d) F. (e) T. (f) T. (g) F.

(b) If each f_n has only finitely many discontinuities, must the same be true of f?

(c) Show that if each f_n has no discontinuities of the second kind, then the same is true of f.

(d) If each f_n has no discontinuities of the first kind, must the same be true of f?

7.3:8. *Examples showing the need for the hypotheses of Theorem 7.13*. (**d**: 2)

Let us examine the need for the various hypotheses in Theorem 7.13 (p.150).

(a) Indicate which examples given by Rudin show that the theorem becomes false (i) if the word ''continuous'' is dropped from hypothesis (b) of the theorem but kept in hypothesis (a); (ii) if condition (c) of the theorem is dropped, and (iii) if the requirement that K be compact is dropped.

(b) Give an example showing that if the word ''continuous'' is deleted from hypothesis (a) of the theorem, but kept in hypothesis (b), the statement also becomes false.

(c) Give an example showing that if the hypothesis that K is compact is replaced by the hypothesis that all f_i are bounded and uniformly continuous, the statement is still false.

7.3:9. *A second uniformity in Theorem 7.11*. (**d**: 2)

In Theorem 7.11, p.149, the condition that the convergence of the f_n be uniform concerns ''uniformity in t'', i.e., it says that for every ε there is an N independent of t with the appropriate property.

Prove that under the conditions of the theorem, the convergence of the $f_n(t)$ to the values A_n is ''uniform in n'', in the sense that for every $\varepsilon > 0$ there exists a $\delta > 0$ (independent of n) such that for every n and every $t \in E$ with $d(t,x) < \delta$, one has $|f_n(t) - A_n| < \varepsilon$.

Suggestion: Choose N such that for $n \geq N$, $|f_n(t) - f_N(t)| < \varepsilon/2$ for all t; then find a δ such that when $d(t,x) < \delta$ and $n \in \{1, \ldots, N\}$, one has $|f_n(t) - A_n| < \varepsilon/2$.

7.3:10. *More on* $\lim f_n(x_n)$. (**d**: 3. >**7:R9**)

As noted in my comment on **7:R9**, the last sentence thereof can be taken ask whether it is true that if a sequence (f_n) of continuous functions on a metric space E and a continuous function f on E have the property that for every sequence of points (x_n) in E which approach a limit x, one has $\lim_{n\to\infty} f_n(x_n) = f(x)$, then (f_n) must converge uniformly to f. Prove that this is so if E is assumed compact.

7.3:11. *Uniform convergence is convergence in a metric even for unbounded functions*. (**d**: 2)

Rudin's observation on p.151, lines 4-6, that uniform convergence is equivalent to convergence in the metric defined on the top line of that page, is necessarily limited to bounded functions, since the supremum of the absolute value of an unbounded function is infinite. Show, however, that if we define $d(f,g) = \min(\|f - g\|, 1)$, then this function gives a metric on the space of all real- or complex-valued functions on any set E, such that uniform convergence of such functions is equivalent to convergence in this metric.

(Remark: Although this metric is of interest for the above reason, it has the disadvantage of not satisfying the law $d(cf, cg) = c\,d(f,g)$ for all positive real numbers c, which is of importance in the study of vector spaces of functions, and which *is* satisfied by the metric $\|f - g\|$ on $\mathscr{C}(X)$.)

7.3:12. *Pointwise convergence is not convergence in any metric*. (**d**: 3. >**7.1:3**, **4.2:10**, **7:R14**)

In contrast to Rudin's observation that uniform convergence of functions in $\mathscr{C}(X)$ is equivalent to convergence in the metric $d(f,g) = \|f - g\|$, we shall show here that there is in general no metric d on $\mathscr{C}(X)$ such that convergence with respect to d is equivalent to pointwise convergence of functions. We shall do this by showing that pointwise convergence does not have the property proved in **7.1:3** for convergence in a metric space, even in the case where all the elements called p_m in that exercise are the same. Specifically, we will construct functions $g_{m,n} \in \mathscr{C}([0,1])$ such that for each m, $\lim_{n\to\infty} g_{m,n} = 0$, but such that there is no sequence of integers (n_k) such that $\lim_{k\to\infty} g_{k,n_k} = 0$.

(Such an example will be obtained here relatively quickly, assuming the earlier exercise **4.2:10**, which was in turn based on Rudin's quite challenging **7:R14**. In **7.3:17** below, we will obtain an example with the same properties with only a little more work, and without relying on more difficult exercises.)

To construct the $g_{m,n}$ recall that in **4.2:10**(c) we found (assuming the result of **7:R14**) a sequence of functions, which we will denote (c_n), such that for every sequence (x_i) of points of $[0,1]$, there exists a point $t\in[0,1]$ such that $c_n(t) = x_n$ for all n. (What we are calling c_n was, in the notation of that exercise, $a\circ b^n$.) Recall also that in Example 7.21 (p.156), Rudin gives a sequence of functions $f_n : [0,1] \to [0,1]$ converging pointwise to 0, but such that each f_n takes the value 1 at some point.

Using the functions c_n and f_n just named, let us, for all $m,\, n \geq 1$, define $g_{m,n} = f_n\circ c_m$.

(a) Show that for each m, $(g_{m,n})$ is a sequence of continuous functions $[0,1] \to [0,1]$ which converges pointwise to the zero function as $n \to \infty$.

(b) Show, however, that given any sequence of positive integers (n_k), one can find $t\in[0,1]$ such that for all k, $g_{k,n_k}(t) = 1$.

(c) Deduce that there is no sequence of positive integers (n_k) such that the sequence of functions g_{k,n_k} converges pointwise to the zero function.

(d) Conclude with the help of **7.1:3** that there is no metric on $\mathscr{C}([0,1])$ such that pointwise convergence of functions in $\mathscr{C}[0,1]$ is equivalent to convergence in this metric.

7.3:13. *Another property of the above example.* (**d**: 4. >**7.3:12** or **7.3:17**)

Let us use the construction of the preceding exercise, or the similar construction of **7.3:17** to show another way that pointwise convergence in $\mathscr{C}([0,1])$ is unlike convergence in a metric space.

Let $(g_{m,n})$ be any family of continuous functions on $[0,1]$ having the properties asserted in **7.3:12** (a) and (b), or in **7.3:17** (a) and (c). Let $\mathscr{A}$ be the set of functions $\{g_{m,n} + 1/m \mid m,n\in J\}$, and let $\mathscr{B}$ be the set of pointwise limits in $\mathscr{C}([0,1])$ of all pointwise convergent sequences of functions in $\mathscr{A}$. Show that there exists a sequence of functions in $\mathscr{B}$ which is pointwise convergent to a function in $\mathscr{C}([0,1])$ that does not lie in $\mathscr{B}$. Why could this not happen if pointwise convergence were convergence with respect to a metric?

7.3:14. *But pointwise convergence on a countable set* is *pointwise convergence in a metric.* (**d**: 3. >**4.2:14**)

Let E be a countable set, and $\mathscr{A}$ the space of all real- or complex-valued functions on E (or any subset thereof). Show that there exists a metric d on $\mathscr{A}$ such that convergence in d is equivalent to uniform convergence of functions. (Suggestion: Use the idea of **4.2:14**.)

7.3:15. *Uniform convergence expressed in terms of uniform continuity.* (**d**: 1, 2)

Let E be a set, (f_n) a sequence of complex-valued functions on E, and f another complex-valued function on E. We shall show that the sequence (f_n) converges (respectively, converges uniformly) to f if and only if a certain function F on a certain metric space X is continuous (respectively, uniformly continuous).

Namely, let $S = \{1/n \mid n\in J\} \cup \{0\} \subseteq R$ (where, following Rudin, we are using J for the positive integers), let $X = E\times S$, meaning the set of ordered pairs (p,s) with $p\in E,\ s\in S$, and let us make X a metric space by defining the distance $d((p,s),(p',s'))$ to be $|s-s'|$ if $p = p'$, and to be 1 otherwise.

(a) Verify that this function is indeed a metric.

Let us now define $F: X \to C$ by letting $F((p,1/n)) = f_n(p)$, and $F((p,0)) = f(p)$.

(b) Show that $f_n \to f$ pointwise if and only if F is continuous, and that $f_n \to f$ uniformly if and only if F is uniformly continuous.

The above exercise puts in visualizable form a way in which the concept of uniform convergence is parallel to that of uniform continuity. I wondered whether I could similarly find a construction whereby the uniform continuity of any function on a metric space could be expressed as the uniform convergence of a sequence of functions on a set. The next exercise gives two such constructions; the one in (a) is simple, but somewhat hoaky; the one in (b) is more natural, but also more complicated. (Actually, the one in (a) can be looked at as a simplified version of the one in (b).)

7.3:16. *Uniform continuity expressed in terms of uniform convergence.* (**d**: 2)

Let X be a metric space, and $F: X \to C$ any function.

(a) Let E denote the set $X \times X$ of all ordered pairs (p,q) with $p,q \in X$, and let us define a sequence (f_n) of functions on E and another function f on E as follows: For each n, and each $(p,q) \in E$, let $f_n(p,q) = F(p)$ if $d(p,q) < 1/n$, and $F(q)$ otherwise. Let $f(p,q) = F(q)$. Show that whatever the choice of F, we will have $f_n \to f$ pointwise, and that this convergence will be uniform if and only if F is uniformly continuous.

(b) Let E' denote the set of those pairs $((p_n), q)$ such that (p_n) is a sequence in X, q is a point of X, and for all n, $d(p_n, q) < 1/n$. For each $m > 0$ let $f_m(((p_n), q)) = F(p_m)$, and let $f(((p_n), q)) = F(q)$. Show that $f_n \to f$ pointwise if and only if F is continuous, and that this convergence is uniform if and only if F is uniformly continuous.

7.3:17. *Another example showing pointwise convergence is not convergence in any metric*. (**d**: 3. >**7.1:3**)

As in **7.3:12** above, we shall construct here a family of functions $g_{m,n}$ whose properties with respect to pointwise convergence would contradict **7.1:3** if pointwise convergence were convergence with respect to a metric; but this time our construction will be self-contained.

For every pair of positive integers m and n, let $g_{m,n} \in \mathscr{C}([0,1])$ be the function such that for each nonnegative integer $k < 2^m$, the values of f on the subinterval $[k2^{-m}, (k+1)2^{-m}] \subseteq [0,1]$ are determined by the formulas

$$\begin{aligned} g_{m,n}(k\,2^{-m} + t\,2^{-m-n}) &= t && \text{for } 0 \le t \le 1, \\ g_{m,n}(k\,2^{-m} + t\,2^{-m-n}) &= 2-t && \text{for } 1 \le t \le 2, \\ g_{m,n}(x) &= 0 && \text{for } x \in [k\,2^{-m} + 2\cdot 2^{-m-n}, (k+1)\,2^{-m}]. \end{aligned}$$

(To get a feel for this definition, you might graph $g_{0,n}$ for the first few values of n (I wrote "$m>0$" above so that the sequences in this exercise would all be indexed by positive integers, but the definition makes sense for $m=0$ as well, and gives the simplest picture), and then go to $m=1$, $m=2$, to see how the behavior varies with m.)

(a) For each m, show that the sequence of functions $g_{m,n}$ $(n=1,2,\ldots)$ converges pointwise to the zero function on $[0,1]$.

Since the sequence of zero functions in turn converges pointwise to the zero function on $[0,1]$, if pointwise convergence were convergence in a metric on $\mathscr{C}([0,1])$, **7.1:3** would imply the existence of a sequence of positive integers $N_1, N_2, \ldots$ such that for any positive integers $n_1, n_2, \ldots$ with $n_m \ge N_m$ for all m, the sequence of functions g_{m,n_m} converged pointwise to 0. To prove the contrary, we will need a description of places where our functions $g_{m,n}$ take on values that are not close to 0.

(b) Prove that if m and n are positive integers, and $x \in [0,1]$ has the property that the digits in the binary expression of x with place-value 2^{-m-j} are 0 for $j = 1, \ldots, n$, while the digit with place-value 2^{-m-n-1} is 1, then $g_{m,n}(x) \ge ½$.

(c) Show with the help of (b) that for any sequence of positive integers $n_1, n_2, \ldots$, there exists $x \in [0,1]$ such that $g_{m,n_m} \ge ½$ for infinitely many values of m.

(d) Deduce that there is no sequence $n_1, n_2, \ldots$ such that the sequence of functions g_{m,n_m} converges pointwise to 0. Conclude using **7.1:3** that pointwise convergence on $\mathscr{C}([0,1])$ is not equivalent to convergence in any metric d on that set.

7.3:18. *Uniform limits of uniformly continuous functions*. (**d**: 2)

Show that if (f_n) is a sequence of complex-valued functions on a metric space X, each of which is uniformly continuous, and if $f_n \to f$ uniformly, then f is also uniformly continuous.

7.3:19. *Locally uniform convergence*. (**d**: 1,2,2,2)

Let us say that a sequence (f_n) of complex-valued functions on a metric space X converges *locally uniformly* to a function f if for every $x \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ and a positive integer N such that for every $n \ge N$, and every y with $d(x,y) < \delta$, one has $|f_n(y) - f(y)| \le \varepsilon$.

(a) Show that uniform convergence implies locally uniform convergence, and locally uniform convergence implies pointwise convergence.

(b) Show by examples that neither of the two preceding implications is reversible. (Looking at part (d) below may help you see what is needed in one of these examples.)

(c) Show that Theorem 7.13 remains true if we delete the assumption that K is compact, while weakening the conclusion by inserting the word ''*locally*'' before ''*uniform*'' on the last line.

(d) Show that on a compact metric space, locally uniform convergence is equivalent to uniform convergence. Deduce Theorem 7.13 from this and part (c) above.

(e) Prove the result of **7.3:10** without the assumption of compactness, but with the conclusion of uniform convergence weakened to locally uniform convergence. With the help of the first statement of (d) above, deduce from this the result of **7.3:10** as stated.

7.3:20. *Locally uniform convergence and composition of functions.* (**d**: 3)

Suppose that $m: X \to Y$ is a continuous function between metric spaces. Recall that if f is a function on Y, then $f \circ m$ denotes the function on X defined by $(f \circ m)(x) = f(m(x))$. Recall also the definition of ''locally uniform'' convergence given in the preceding exercise.

(a) Show that if f and f_n $(n \geq 1)$ are continuous complex-valued functions on Y, and if $f_n \to f$ locally uniformly, then $f_n \circ m \to f \circ m$ locally uniformly.

(b) Show by example that the result of (a) becomes false if the word ''locally'' is deleted in both places.

(c) Show, on the other hand, that the statement shown to be false in (b) becomes true again if the assumption on $m: X \to Y$ is strengthened from ''continuous'' to ''uniformly continuous''.

7.4. UNIFORM CONVERGENCE AND INTEGRATION. (pp.151-152)

Relevant exercises in Rudin:

7:R10. *Analyzing the discontinuities of a messy function.* (**d**: 3)

Here ''Find all discontinuities of f'' means ''Find all points where f is discontinuous''.

I also recommend looking at **7:R21** at this point. Though the concepts of ''algebra of functions'', ''separating points'' and ''vanishing nowhere'' have not yet been defined, the key part of this problem is the verification that for every function in the uniform closure of $\mathscr{A}$, the integral shown is zero, and that is a nice application of the results of this section. When you get to the last theorem of this chapter, this example will show that a curious hypothesis in that result, called ''self-adjointness'', is really needed.

Exercises not in Rudin:

7.4:0. *Say whether the following statement is true or false.*

(a) If a sequence of continuous real-valued functions f_n on $[-1,1]$ converges uniformly to a function f, then the sequence of numbers $\int_{-1}^{1} f_n\,dx$ converges to the number $\int_{-1}^{1} f\,dx$.

7.4:1. *Pointwise convergent series of functions cannot always be integrated term-by-term.* (**d**: 1)

Show by example that the corollary to Theorem 7.16 (p.152) becomes false if ''converging uniformly'' is replaced by ''converging pointwise''. (Suggestion: Find an example in Rudin showing that Theorem 7.16 itself becomes false if ''uniformly'' is replaced by ''pointwise'', and show how the functions in that example can be made the partial sums in a series.)

7.5. UNIFORM CONVERGENCE AND DIFFERENTIATION. (pp.152-154)

Relevant exercise in Rudin:

7:R7. *Another counterexample involving uniform convergence and differentiation.* (**d**: 1)

Example 7.5 (p.146) already shows that ''uniform convergence doesn't respect differentiation''. So after doing this exercise, you should ask yourself what this exercise shows that that example doesn't. In other words, what statement might one have hoped to prove despite Example 7.5, which this example shows is false? You should also note how the true fact about uniform convergence and differentiation proved in this section differs from the statements that these counterexamples show to be false.

Exercises not in Rudin:

7.5:0. *Say whether each of the following statements is true or false.*

(a) If a sequence of differentiable functions f_n on $[-1,1]$ converges uniformly to a differentiable function f, then the functions f_n' converge pointwise to f'.

(b) If (f_n) is a sequence of differentiable functions on $[-1,1]$, and f a differentiable function on that interval such that the functions f_n' converge uniformly to f', then the functions f_n converge uniformly to f.

(c) If (f_n) is a sequence of differentiable functions on $[a,b]$ such that the sequence of derivatives f_n' converges uniformly, then the sequence of functions g_n defined by $g_n(x) = f_n(x) - f_n(a)$ converges uniformly.

(d) If f is uniformly continuous on $[a,b]$, then f is differentiable at all but at most countably points of $[a,b]$.

7.5:1. *Rudin's nowhere-differentiable function has non-rectifiable graph.* (**d**: 3)

Let f be the nowhere differentiable function constructed in the proof of Theorem 7.18 (display (37), p.154). Show that the graph of f on the interval [0,1], i.e., the curve γ on [0,1] given by $\gamma(x) = (x, f(x))$, is not rectifiable. (Suggestion: For m a positive integer, partition [0,1] into $2\cdot 4^m$ equal segments, and show that of any two successive segments, at least one has the property that $f(x_i) - f(x_{i-1})$ is ''large'' in an appropriate sense.)

7.5:2. *Theorem 7.17 for series.* (**d**: 1)

State a corollary to Theorem 7.17 giving a sufficient condition for a series of differentiable functions to sum to a differentiable function, with a description of the derivative of that function.

7.6. EQUICONTINUOUS FAMILIES OF FUNCTIONS. (pp.154-158)

Relevant exercises in Rudin:

7:R1. *Uniform convergence and boundedness imply uniform boundedness.* (**d**: 2)

7:R11. *Conditions for* $\Sigma f_n g_n$ *to converge uniformly.* (**d**: 2)

7:R12. *Dominated convergence for improper integrals.* (**d**: 3)

Improper integrals are defined in **6:R7** and **6:R8**; the definitions are needed to do this exercise, but the results of those exercises are not.

Note that the integrals of this problem are improper in two ways: Not only is the upper limit of integration $+\infty$, but the functions are not assumed to be Riemann-integrable on intervals $[0,T]$, but only on intervals $[t,T]$; so the lower limit of integration 0 also involves taking a limit. However, these complications are independent of one another, because one can write $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$, prove the desired result for the two integrals separately, and get the result Rudin asserts by summing. Since Rudin has not formally defined doubly improper integrals, you shouldn't worry about how to prove the formula $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$; for the sake of this exercise, you can regard it as a definition.

7:R13. *Helley's selection theorem: finding a convergent subsequence of a sequence of increasing functions.* (**d**: 3)

In part (b), ''$f_{n_k} \to f$ uniformly on compact sets'' means that for every compact set $K \subseteq R$, the restrictions of the functions f_{n_k} to K converge uniformly to the restriction of f to K.

7:R15. *For what f's is* $\{f(nt)\}$ equicontinuous? (**d**: 1)

Rudin asks ''What conclusion can you draw?'' If you don't know where to begin, look for examples of functions f with this property. The examples that you find will be very restricted; see whether you can

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Answer to True/False question **7.4:0.** (a) T.

show that the restrictions in the examples you discover must be satisfied by all examples.

7:R16. *For an equicontinuous sequence of functions on a compact set, pointwise convergence is uniform.* (**d**: 3)

7:R17. *Uniform convergence and equicontinuity for mappings into general metric spaces.* (**d**: ?)

Like **3:R15**, this exercise asks the student to generalize a large number of results in the chapter to a more general context. As with that exercise, it is not clear whether it would be reasonable to assign this as homework, or if one did, what instructions to give; but the exercise is certainly worth thinking about.

7:R18. *A uniformly convergent subsequence of a sequence of integrals.* (**d**: 1)

7:R19. *Conditions for a subset of* $\mathscr{C}(K)$ *to be compact.* (**d**: 2. > **2:R26**)

7:R25. *Existence of a solution to a differential equation with initial conditions.* (**d**: 3)

One may ask why a result on finding convergent subsequences of not-necessarily-convergent sequences should be needed to prove such a result. The method of attacking the differential equation that is described uses successive approximations; won't that method actually converge?

In general, no! At the end of **5:R27**, Rudin noted an example of a differential equation with nonunique solution for given initial conditions. This indicates an ''instability'' in such structures, which has the consequence that when we use the method of this exercise to approximate solutions, our successive steps may approximate *different* solutions, so that the sequence of approximations may not converge, and we may really have to pass to a subsequence to get a solution as our limit.

Note that at the start of his Hint, where Rudin says ''Let f_n ...'', you are to show that such a function exists. A few lines later, where after defining $\Delta_n(t)$ he adds ''except ...'', he means that for $t = x_i$, one defines $\Delta_n(t) = 0$. (The preceding definition is not applicable when $t = x_i$ because $f_n'(x_i)$ is generally undefined; so the arbitrary value 0 is used to make $\Delta_n(t)$ defined.)

7:R26. *The corresponding result for a* system *of differential equations.* (**d**: 3. > **7:R25**)

Rudin's hint says ''Use the vector-valued version of Theorem 7.25''. He means that you should prove such a version and use it. The vector-valued result is not hard to prove from the theorem as given.

Exercises not in Rudin:

7.6:0. *Say whether each of the following statements is true or false.*

(a) The set of functions $\{x^n \mid n = 1, 2, ...\}$ on $[-1, 1]$ is uniformly bounded.

(b) The set of functions $\{x^n \mid n = 1, 2, ...\}$ on $[0, 2]$ is uniformly bounded.

(c) Every uniformly bounded family of continuous functions on a compact set is equicontinuous.

(d) If $\mathscr{B}$ is a family of differentiable functions on R such that the set $\{f' \mid f \in \mathscr{B}\}$ is uniformly bounded, then $\mathscr{B}$ is equicontinuous.

7.6:1. *When do boundedness at one point and equicontinuity imply pointwise boundedness?* (**d**: 4, 1, 4, 2, 2)

(a) Show that the following two conditions on a metric space X are equivalent:

(i) Every equicontinuous family of functions on X whose values at one point are bounded is pointwise bounded. (I.e., if $\mathscr{F}$ is an equicontinuous family of functions on X, and if there exists $x \in X$ such that $\{f(x) \mid f \in \mathscr{F}\}$ is bounded, then $\{f(y) \mid f \in \mathscr{F}\}$ is bounded for every $y \in X$.)

(ii) For every two points $x, y \in X$, and every $\delta > 0$, there exist points $x_0, \ldots, x_n$ such that $x = x_0$, $y = x_n$, and $d(x_{i-1}, x_i) < \delta$ for $i = 1, \ldots, n-1$.

(b) Show that $X = [0,1] \cup [2,3]$ does not satisfy the above equivalent conditions (i) and (ii).

(c) Show that every connected metric space satisfies the above equivalent conditions.

(d) Show that if a metric space X satisfies the above equivalent conditions, then so does every dense subset of X.

Answers to True/False question **7.5:0.** (a) F. (b) F. (c) T. (d) F.

(e) Use the results of (c) and (d) to find an open subset of R which satisfies the equivalent conditions of (a), but which is not connected.

7.6:2. *Proving Example 7.20 without using a result from Chapter 11.* (**d**: 2)

In Example 7.20, Rudin shows that the sequence of functions $f_n = \sin nx$ on the interval $[0, 2\pi]$ has no pointwise convergent subsequence, but to do so, he calls on a result in Chapter 11. Below, you will prove this result using only material from Chapters 1-4, and the facts that $\sin x$ is a continuous function which takes on the value 1 at $x = \pi/2$ and the value -1 at $x = 3\pi/2$, and satisfies $\sin(x+2\pi) = \sin x$ for all x. When we speak of an ''interval $[a, b]$'' below, we shall understand this to entail $a < b$.

(a) Show that for any interval $[a, b] \subseteq R$ there exists an integer N such that for all $n > N$ the function $\sin nx$ takes on both the values 1 and -1 at points of $[a, b]$.

(b) Deduce that for any infinite set S of positive integers and any interval $[a, b]$, there exists an $n \in S$, and subintervals $[a', b']$ and $[a'', b'']$ of $[a, b]$, such that the function $\sin nx$ has value everywhere $\geq \frac{1}{2}$ on $[a', b']$, and has value everywhere $\leq -\frac{1}{2}$ on $[a'', b'']$.

(c) Deduce from (b) that for any infinite set S of positive integers one can find a point x and a sequence $n_1 < n_2 < \ldots$ in S such that the real numbers $\sin n_1 x$, $\sin n_2 x$, ... are alternately $\geq \frac{1}{2}$ and $\leq -\frac{1}{2}$.

(d) Conclude that the sequence of functions $\sin nx$ $(n = 1, 2, \ldots)$ has no pointwise convergent subsequence.

7.7. The Weierstrass Theorem, and a corollary (beginning of Rudin's section THE STONE-WEIERSTRASS THEOREM). (pp.159-161)

Relevant exercises in Rudin:

7:R20. *A continuous function on* $[0,1]$ *is determined by its moments.* (**d**: 2)

The integral on the left-hand side of the displayed equation of this exercise is called the nth *moment* of the function f. Add to this exercise ''Deduce that if g and h are continuous functions on $[0,1]$ such that for every n, the nth moments of g and of h are the same, then $g = h$''. (This is the meaning of the title I have given this exercise.) In Rudin's Hint for the exercise, $f^2(x)$ in the integral should be changed to $|f(x)|^2$.

7:R22. *Every integrable function is* L^2*-approximable by polynomials.* (**d**: 3. >**6:R12**)

Suggestion: Use the exercise Rudin refers to parenthetically at the end, together with a result from this section.

7:R23. *An explicit algorithm for uniformly approximating* $|x|$ *by polynomials.* (**d**: 3)

This gives a direct proof of Corollary 7.27, which is the only consequence of the Weierstrass Theorem that Rudin will use in proving the Stone-Weierstrass Theorem. (As noted above, **7.3:1** gives another proof of this result.)

Exercises not in Rudin:

7.7:0. *Say whether the following statement is true or false.*

(a) A complex-valued function on $[-1, 1]$ is continuous if and only if it can be written as the uniform limit of a sequence of polynomial functions.

7.7:1. *The Weierstrass Theorem fails for functions on the whole line.* (**d**: 2)

(a) Show that the only polynomials which, as functions on R, are bounded, are the constant functions. (Suggestion: Use results of Chapter 3.)

(b) Deduce that, in contrast with Theorem 7.26, if a sequence of polynomials P_n converges uniformly on the whole real line R to a function f, then f is itself a polynomial.

Answers to True/False question **7.6:0.** (a) T. (b) F. (c) F. (d) T.

7.7:2. *A modified Weierstrass theorem that does work for functions on all of* R. (**d**:2)

Suppose f is a continuous complex-valued function on the real line. Show that there exists a sequence of polynomials P_n such that for each finite interval $[a,b]$, the polynomials P_n converge uniformly to f on $[a,b]$.

7.8. Algebras of Functions, Uniform Closure, and Separation of Points (middle of Rudin's section **THE STONE-WEIERSTRASS THEOREM).** (pp.161-162)

Relevant exercises in Rudin: None

Exercises not in Rudin:

7.8:0. *Say whether each of the following statements is true or false.*

(a) For every metric space X, $\mathscr{C}(X)$ is an algebra of functions on X.

(b) If $\mathscr{A}$ and $\mathscr{B}$ are uniformly closed algebras of functions on a set E, then $\mathscr{A}\cap\mathscr{B}$ is also a uniformly closed algebra of functions on E.

(c) The set of all monotonically increasing real-valued functions α on an interval $[a,b]$ is an algebra.

(d) The set of all monotonic real-valued functions α (increasing and decreasing) on an interval $[a,b]$ is an algebra.

(e) If a family of functions $\mathscr{A}$ on a set E separates points, then so does every family of functions on E containing $\mathscr{A}$.

(f) If a family of functions $\mathscr{A}$ on a set E separates points, then so does every subset of $\mathscr{A}$.

(g) If a family of functions $\mathscr{A}$ on a set E vanishes at no point of E, then so does every family of functions on E containing $\mathscr{A}$.

(h) If a family of functions $\mathscr{A}$ on a set E vanishes at no point of E, then the same is true of every subset of $\mathscr{A}$.

7.8:1. *Equicontinuous algebras are mostly uninteresting*. (**d**:2,1,2,4)

This exercise will show that ''equicontinuous'' is not, in general, an interesting condition to impose on *algebras* of functions, though as (b) shows, there are some nontrivial examples.

(a) Show that there is no equicontinuous algebra of real-valued functions on $[0,1]$ which separates points.

(b) Show that the algebra of all real-valued functions on Z is equicontinuous and separates points.

(c) Let $X\subseteq\{1/n \mid n=1, 2, 3, ...\}$. Does there exist an equicontinuous algebra of real-valued functions on X which separates points?

(d) Give a simple characterization of the class of metric spaces X such that there exists an equicontinuous algebra of real-valued functions on X which separates points.

7.8:2. *Transporting algebras of functions from one metric space to another*. (**d**:1,1,2,3,3)

Throughout this exercise, let $m: X\to Y$ be a continuous map of metric spaces. Recall that if f is a function on Y, then $f\circ m$ denotes the function on X defined by $(f\circ m)(x)=f(m(x))$.

(a) Show that if (f_n) is a sequence of functions on Y that converges uniformly to a function f, then the sequence of functions $(f_n\circ m)$ on X converges uniformly to $f\circ m$.

In the remaining parts, let $\mathscr{A}$ be an algebra of continuous functions on X, and let $\mathscr{B}$ be an algebra of continuous functions on Y. Let $m^*(\mathscr{B})=\{f\circ m \mid f\in\mathscr{B}\}$, and let $m_*(\mathscr{A})$ denote the set of all continuous functions f on Y such that $f\circ m\in\mathscr{A}$. To avoid confusion with complex conjugation, let us use cl for uniform closure; e.g., $\mathrm{cl}(\mathscr{A})$ will denote the uniform closure of $\mathscr{A}$.

(b) Show that $m^*(\mathscr{B})$ is an algebra of continuous functions on X, and that $m_*(\mathscr{A})$ is an algebra of continuous functions on Y.

Answer to True/False question **7.7:0.** (a) T.

(c) Show that $m^*(\mathrm{cl}(\mathscr{B})) \subseteq \mathrm{cl}(m^*(\mathscr{B}))$ and that $m_*(\mathrm{cl}(\mathscr{A})) \supseteq \mathrm{cl}(m_*(\mathscr{A}))$.

(d) Show that the reverses of the above two inequalities do not hold in general. (Suggestions: For the first inequality, let m be the inclusion map $[0,1] \to R$, i.e., the map defined by $m(x) = x$, and $\mathscr{B}$ the algebra of all polynomial functions on R. For the second inequality, let $X = [-1,1]$, $Y = [0,1]$, $m(x) = \max(0, x)$, and let $\mathscr{A}$ be the algebra of all polynomial functions on $[-1,1]$. You may assume **7.7:1** whether or not you did it, and you may assume the fact that the only polynomial $p(x)$ such that the equation $p(x) = 0$ has infinitely many solutions is the zero polynomial.)

(e) What implications, if any, hold between the statements "$\mathscr{A}$ is uniformly closed" and "$m_*(\mathscr{A})$ is uniformly closed"? Between "$\mathscr{B}$ is uniformly closed" and "$m^*(\mathscr{B})$ is uniformly closed"?

(There are four possible implications – one each way for each of the indicated pairs of statements. For full credit you need to give, for each of these four implications, either a proof that it is true, or an example showing that it is false. In doing this, you may assume any of the previous parts of this exercise, whether you did them or not.)

7.8:3. *The uniform closure of the algebra of Laurent polynomials.* (**d**:4. >**7.7:1**)

A *Laurent polynomial* means a function which can be written in the form $f(x) = \Sigma_{n=-N}^{N}\ a_n z^n$, where $N \geq 0$ and $a_{-N}, \ldots, a_N$ are constants. (An example is $z^{-2} + 3z^{-1} - 7 + z^2 + 5z^3$, which we can get by taking $N = 3$ and letting $a_{-3} = 0$, $a_{-2} = 1$, ... $a_3 = 5$.) A Laurent polynomial can be evaluated at any *nonzero* value of x; in particular, given any subset E of R not containing the point 0, the Laurent polynomials with real coefficients yield an algebra of real-valued continuous functions on E. Given a Laurent polynomial $\Sigma_{n=-N}^{N}\ a_n z^n$, let us call $\Sigma_{n=0}^{N}\ a_n z^n$ its "polynomial part" and $\Sigma_{n=-N}^{-1}\ a_n z^n$ its "negative-exponent part".

(a) Let E be the half-open interval $(0,1]$, and suppose that (f_k) is a sequence of Laurent polynomials which, regarded as a sequence of functions on E, converges uniformly. Show that if we write each f_k as $g_k + h_k$, where g_k is its polynomial part and h_k its negative-exponent part, then all but finitely many of the functions h_k are equal, and the sequence of functions (g_k) converges uniformly.

(b) Deduce that the uniform closure of the algebra of Laurent polynomial functions on $(0,1]$ consists of all functions which can be written as the sum of a continuous function and a "negative-exponent Laurent polynomial", i.e., a function of the form $\Sigma_{n=-N}^{-1}\ a_n z^n$.

(c) Deduce that the uniform closure of the algebra of Laurent polynomial functions on $(0,1]$ is not an algebra. This shows that a certain word cannot be omitted from the statement of Theorem 7.29 – which word?

7.9. The Stone-Weierstrass Theorem (end of Rudin's section **THE STONE-WEIERSTRASS THEOREM**). (pp.162-165)

Relevant exercise in Rudin:

7:R21. *The need for self-adjointness in the complex Stone-Weierstrass Theorem.* (**d**:2)

(A more general version of this result is **7.9:7** below.)

Exercises not in Rudin:

7.9:0. *Say whether each of the following statements is true or false.*

(a) If $\mathscr{A}$ is an algebra of real-valued continuous functions on $[0,1]$, and the function $f(x) = x+1$ belongs to $\mathscr{A}$, then the uniform closure of $\mathscr{A}$ is the algebra of all real-valued continuous functions on $[0,1]$.

(b) If $\mathscr{A}$ is an algebra of real-valued continuous functions on $[0,1]$, and the function $f(x) = x-1$ belongs to $\mathscr{A}$, then the uniform closure of $\mathscr{A}$ is the algebra of all real-valued continuous functions on $[0,1]$.

Answers to True/False question **7.8:0.** (a) T. (b) T. (c) F. (d) F. (e) T. (f) F. (g) T. (h) F.

(c) The uniform closure of any algebra of polynomials on $[-1,1]$ consists of polynomials.

(d) If $\mathcal{A}$ is an algebra of real-valued functions on $[0,1]^2$ which contains the functions defined by $f(x,y)=e^x$ and $g(x,y)=(y+1)^{-1}$, then the uniform closure of $\mathcal{A}$ contains the function $(x+1)^y$.

(e) The set of all polynomial functions in one variable with complex coefficients, regarded as an algebra of functions on $[0,1]$, is self-adjoint.

(f) Let $D=\{z\in C \mid |z|\leq 1\}$. The set of all polynomial functions in one variable with complex coefficients, regarded as an algebra of functions on D, is self-adjoint.

(g) If K is a compact metric space, then the only uniformly closed self-adjoint algebra of complex-valued continuous functions on K which separates points and vanishes at no point of K is $\mathcal{C}(K)$.

7.9:1. *Some examples on which to try out the Stone-Weierstrass Theorem.* (**d**: 2)

For each of the following sets $\mathcal{A}_i$ of continuous functions (some real and some complex-valued), determine whether the uniform closure comprises *all* continuous functions (real or complex as the case may be) on the given domain. In each of these cases, you will either be able to show that it does by the Stone-Weierstrass Theorem, or show that it does not by finding a continuous function which is not a uniform limit of functions in the given set. (The next exercise will ask you to show that there are sets of functions whose uniform closures give all continuous functions, but which do not satisfy the conditions of the Stone-Weierstrass Theorem; but no such examples occur in this exercise.)

In cases where the uniform closure consists of all continuous functions, an answer ''Yes'' is all that you need to write down. In each case of the opposite sort, give an example of a function not in the uniform closure of $\mathcal{A}_i$, and state at least one hypothesis of the Stone-Weierstrass Theorem which fails to hold for that set. If the condition that fails is that $\mathcal{A}_i$ be an algebra, state one of the properties defining an algebra which is not satisfied by $\mathcal{A}_i$. For your own sake, you should also be able to *show* that the function you give is not in the uniform closure; but you are not asked for this verification in your homework.

(a) The set $\mathcal{A}_1$ of all complex-valued polynomial functions f on $[0,1]$ which satisfy $f(0)=f(1)$.

(b) The set $\mathcal{A}_2$ of all real-valued polynomial functions f on $[0,1]$ that satisfy $f'(½)=0$.

(c) The set $\mathcal{A}_3$ of all continuous real-valued functions f on R such that $\lim_{x\to+\infty} f(x)$ exists.

(d) The set $\mathcal{A}_4$ of all complex-valued polynomial functions f on $[0,1]$ that satisfy $f(1)=\overline{f(0)}$.

(e) The set $\mathcal{A}_5$ of all real-valued continuous functions f on $[0,1]$ that satisfy $f(0)+f(½)+f(1)=0$.

(f) The set $\mathcal{A}_6$ of all functions on $[0,1]$ of the form $p(x)$, where p is a real-valued polynomial which is divisible by $x-1$.

(g) The set $\mathcal{A}_7$ of all functions on $[0,1]$ of the form $p(x)$, where p is a real-valued polynomial which is divisible by $x-2$.

(h) The set $\mathcal{A}_8$ of all continuous real-valued functions f on $[0,1]$ that satisfy

$$(\exists\ \varepsilon>0)\ (\forall\, x\in[0,\varepsilon])\ \ f(x)=f(0).$$

7.9:2. *The hypothesis of the Stone-Weierstrass Theorem is sufficient but not necessary.* (**d**: 2)

Give an example of a set $\mathcal{A}$ of real-valued functions on a metric space K which does not satisfy all the hypotheses of Theorem 7.32, but such that the uniform closure of $\mathcal{A}$ does consist of all continuous real-valued functions on K.

7.9:3. *An even or odd function is uniformly approximable by even or odd polynomials.* (**d**: 2)

(a) Let $\mathcal{A}$ be the algebra of all even polynomial functions on $[-1,1]$ (polynomial functions p satisfying $p(-x)=p(x)$). Show that the uniform closure of $\mathcal{A}$ consists of all even continuous functions. (The easier direction is ''$\subseteq$''. Suggestion for ''$\supseteq$'': Apply the Stone-Weierstrass Theorem to the restrictions of these functions to $[0,1]$.)

(b) Let $\mathcal{A}$ be the set of all odd polynomial functions on $[-1,1]$ (polynomial functions p satisfying $p(-x)=-p(x)$). Show that the uniform closure of $\mathcal{A}$ consists of all odd continuous functions.

(Suggestion for ‘‘$\supseteq$’’: Approximate such a function f by polynomials p_n, break each p_n into the sum of an odd and an even polynomial, and show that the odd summands also approximate f.)

7.9:4. *Functions with value zero at* 0 *are uniformly approximable by polynomials with value zero at* 0. (**d**: 2)

Let $\mathcal{A}$ be the algebra of all polynomial functions p on $[-1,1]$ satisfying $p(0) = 0$. Show that the uniform closure of $\mathcal{A}$ consists of all continuous functions f satisfying $f(0) = 0$. (As in the previous exercise, the easier direction is ‘‘$\subseteq$’’. Suggestion for ‘‘$\supseteq$’’: See Rudin’s proof of Corollary 7.27.)

7.9:5. *Intersections of uniform closures, and uniform closures of intersections*. (**d**: 4, 2)

Let $\mathcal{A}$ be the set of restrictions to $[0,1]$ of even polynomial functions, i.e., polynomial functions satisfying $p(-x) = p(x)$, and let $\mathcal{B}$ be the set of restrictions to $[0,1]$ of polynomial functions satisfying $p(2-x) = p(x)$.

(a) Show that $\mathcal{A} \cap \mathcal{B}$ consists only of constant functions.

(b) Let us write $\mathrm{cl}(\mathcal{A})$ instead of $\overline{\mathcal{A}}$ for the uniform closure of $\mathcal{A}$, to avoid confusion with complex conjugation. Show that $\mathrm{cl}(\mathcal{A}) = \mathrm{cl}(\mathcal{B}) = \mathcal{C}([0,1])$. Deduce that $\mathrm{cl}(\mathcal{A}) \cap \mathrm{cl}(\mathcal{B}) \neq \mathrm{cl}(\mathcal{A} \cap \mathcal{B})$.

7.9:6. *Uniform closures of algebras of continuous real functions not satisfying the hypotheses of the Stone-Weierstrass Theorem*. (**d**: 1, 2)

Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact set K, but let us *not* assume that $\mathcal{A}$ separates points or vanishes at no point of K. Rather, given any continuous real-valued function f on K, let us say that f ‘‘separates no points not separated by $\mathcal{A}$’’ if for all $x, y \in K$, we have

$$((\forall\ h \in \mathcal{A})\ h(x) = h(y)) \ \Rightarrow\ f(x) = f(y),$$

and let us say that f ‘‘vanishes wherever $\mathcal{A}$ vanishes’’ if for all $x \in K$, we have

$$((\forall\ h \in \mathcal{A})\ h(x) = 0) \ \Rightarrow\ f(x) = 0.$$

(You might find these conditions easier to think about in contrapositive form:

$f(x) \neq f(y) \ \Rightarrow\ (\exists\ h \in \mathcal{A})\ h(x) \neq h(y)$, respectively, $f(x) \neq 0 \ \Rightarrow\ (\exists\ h \in \mathcal{A})\ h(x) \neq 0$.)

Then I claim that

The uniform closure of $\mathcal{A}$ consists of all continuous real-valued functions f that separate no points not separated by $\mathcal{A}$, and vanish wherever $\mathcal{A}$ vanishes.

(a) Show (by arguments and/or quoting results from Rudin) that all functions f in the uniform closure of $\mathcal{A}$ are indeed continuous, separate no points not separated by $\mathcal{A}$, and vanish wherever $\mathcal{A}$ vanishes.

To prove the converse, suppose f is a continuous real-valued function on K which separates no points not separated by $\mathcal{A}$ and vanishes wherever $\mathcal{A}$ vanishes. We must show that f is uniformly approximable by members of $\mathcal{A}$. This can be done by a small change in the proof of Theorem 7.32.

Steps 1 and 2 of that proof do not use anything about separating points or not vanishing, and so need no change. In the *statements* of Steps 3 and 4, the only change needed is to add, after ‘‘*a real function f, continuous on K*’’, the words ‘‘*which separates no points not separated by $\mathcal{A}$, and vanishes wherever $\mathcal{A}$ vanishes*’’. All assertions in the proofs of those two steps then become true, except for the first sentence of the proof of Step 3, which is used to justify the second sentence. So –

(b) Prove that second sentence (the one beginning ‘‘Hence’’ and ending with display (55)) under the above hypotheses. (You will need to consider different cases, depending on whether $\mathcal{A}$ separates x and y, and whether it vanishes on one or both of these points.)

This completes the proof of the result stated in italics above.

(If you are careful, you will see the need for one small condition in the definition of an algebra that Rudin accidentally omitted: That it be *nonempty*, i.e., contain at least one function. Assume this.)

7.9:7. *The Stone-Weierstrass theorem fails for non-self-adjoint algebras of complex-valued functions*. (**d**: 3)

Let K be any compact subset of the complex plane which contains the origin 0 and the unit circle

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Answers to True/False question **7.9:0.** (a) T. (b) F. (c) F. (d) T. (e) T. (f) F. (g) T.

$\{z\in C \mid |z| = 1\}$ (for instance, the unit disk $D = \{z\in C \mid |z| \leq 1\}$), and let $\mathscr{A}$ be the algebra of *polynomial functions* on K, i.e., functions f of the form $f(z) = \Sigma_{n=0}^{N} a_n z^n$ where $a_0, \ldots, a_N \in C$.

(a) Show that $\mathscr{A}$ separates points of K and vanishes nowhere on K.

(b) Show that every $f\in\mathscr{A}$ satisfies

$$f(0) = \tfrac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, dt.$$

(Hint: First verify this for each of the functions z^n. In doing so, you may assume that exponentials of imaginary numbers, defined by formula (32) on p.112, satisfy the familiar formula $(e^z)^n = e^{nz}$.)

(c) Show that if (f_n) is a sequence of continuous functions on K which each satisfy the equation of (b), and if this sequence converges uniformly to a function f, then f also satisfies that equation.

(d) Deduce that the uniform closure of the algebra $\mathscr{A}$ is not the whole algebra $\mathscr{C}(K)$. (This phenomenon, in particular, the above integral formula, can be regarded as a tip of the iceberg of the subject of Complex Analysis.)

(e) Deduce from parts (a) and (b) the result of **7:R21**.